THE CAMBRIDGE

MATHEMATICAL JOURNAL.

VOL. IV.] MAY, 1845. [No. XXIV.

I.—ON THE PROOF OF THE PROPOSITION THAT $(Mx + Ny)^{-1}$ IS AN INTEGRATING FACTOR OF THE HOMOGENEOUS DIFFERENTIAL EQUATION $M + N\dfrac{dy}{dx} = 0$.

By G. G. STOKES, M.A. Fellow of Pembroke College.

A FALLACIOUS proof is sometimes given of this proposition, which ought to be examined. The substance of the proof is as follows.

Let us see whether it is possible to find a multiplier V, a homogeneous function of x and y, which shall render $Mdx + Ndy$ an exact differential. Let M and N be of n, and V of p dimensions; let

$$dU = V(Mdx + Ndy) \ldots\ldots\ldots (1);$$

$$\left.\begin{array}{l}\text{then, on properly choosing the arbitrary constant in } U,\\ U \text{ will be a homogeneous function of } n + p + 1 \text{ dimen-}\\ \text{sions,}\end{array}\right\} (A),$$

whence, by a known theorem,

$$(n + p + 1)\,U = x\frac{dU}{dx} + y\frac{dU}{dy} = V(Mx + Ny) \ldots (2);$$

therefore, dividing (1) by (2),

$$\frac{dU}{(n + p + 1)\,U} = \frac{Mdx + Ndy}{Mx + Ny};$$

and the first side of this equation being an exact differential, it follows that the second side is so also, and consequently that $(Mx + Ny)^{-1}$ is an integrating factor.

Now the factor so found is of $-n-1$ dimensions; so that the first side of (2) is zero. In fact, we shall see that the statement (A) is not true as applied to the case in question, unless $Mx + Ny = 0$.

The general form of a function of x of n dimensions is Ax^n. The general form of a homogeneous function of x and y of n dimensions is $x^n \psi\left(\frac{y}{x}\right)$. The integral of the first is in general $\frac{Ax^{n+1}}{n+1}$, omitting the arbitrary constant; and consequently the dimensions of the function are increased by unity by integration. But in the particular case in which $n = -1$, the integral is $A \log x$, which is not a quantity of o dimensions, at least according to the definition just given, *according to which definition only* is the proposition with reference to homogeneous functions assumed in (2) true. Let us now examine in what cases U will be of $n+p+1$ dimensions.

Putting $M = M_0 x^n$, $N = N_0 x^n$, $y = xz$, M_0 and N_0 will be functions of z alone, and we shall have

$$Mdx + Ndy = x^n \{(M_0 + N_0 z)\, dx + N_0 x\, dz\}.$$

If $M_0 + N_0 z = 0$, i.e. if $Mx + Ny = 0$, we see that x^{-n-1} will be an integrating factor. The integral, being a function of z, will be of 0 dimensions, and both sides of (2) will be zero.

If $Mx + Ny$ is not equal to 0, we may multiply and divide by $(M_0 + N_0 z)\, x$, and we have

$$Mdx + Ndy = x^{n+1} (M_0 + \mathrm{N}_0 z) \left(\frac{dx}{x} + \frac{N_0 dz}{M_0 + N_0 z}\right).$$

Hence we see that $\{x^{n+1}(M_0 + N_0 z)\}^{-1}$ or $(Mx + Ny)^{-1}$ is an integrating factor. For this factor we have

$$U = \log(x) + \phi\left(\frac{y}{x}\right),$$

ϕ denoting the function arising from the integration with respect to z.

In this case we have $x\dfrac{dU}{dx} + y\dfrac{dU}{dy} = 1$, not $= 0$.

It may be of some interest to enquire in what cases an exact differential of any number of independent variables, in which the differential coefficients are homogeneous functions of n dimensions, has an integral which is a homogeneous function of $n+1$ dimensions.

Let $dU = Mdx + Ndy + Pdz + \dots$ be the exact differential. Let $y = y'x$, $z = z'x \dots$, $M = M_0 x^n$, $N = N_0 x^n \dots$, so that M_0, $N_0 \dots$ are functions of y', $z' \dots$ only; then

$$dU = x^n \{(M_0 + N_0 y' + P_0 z' \dots)\, dx + (N_0 dy' + P_0 dz' \dots)\, x\}.$$

First, suppose the coefficient of dx in this equation to be zero, or $Mx + Ny + Pz \dots = 0$; then the expression for dU cannot be an exact differential unless $n = -1$. In this case U will be a function of $y', z' \dots$, and will therefore be a homogeneous function of $n+1$ or 0 dimensions.

Secondly, suppose the coefficient of dx not to be zero; then

$$dU = x^{n+1}(M_0 + N_0 y' \dots)\left(\frac{dx}{x} + \frac{N_0 dy' + P_0 dz' \dots}{M_0 + N_0 y' + P_0 z' \dots}\right)$$

$$= (Mx + Ny + Pz \dots)\left(\frac{dx}{x} + \frac{N_0 dy' + P_0 dz' \dots}{M_0 + N_0 y' + P_0 z' \dots}\right) \dots (3).$$

Now I say that $\dfrac{N_0 dy' + P_0 dz' \dots}{M_0 + N_0 y' + P_0 z' \dots}$ is the exact differential of a function of the independent variables $y', z' \dots$, or, taking $y, z \dots$ for the independent variables instead of $y', z' \dots$, x being supposed constant, and putting for $M_0, N_0 \dots$ their values, that $\dfrac{Ndy + Pdz + \dots}{Mx + Ny + Pz \dots}$ is an exact differential.

For, putting $Mx + Ny + Pz \dots = D$, in order that the quantity considered should be an exact differential, it is necessary and sufficient that the system of equations of which the type is $\dfrac{d\frac{N}{D}}{dz} = \dfrac{d\frac{P}{D}}{dy}$ should be satisfied. This equation gives

$$D\left(\frac{dN}{dz} - \frac{dP}{dy}\right) + P\frac{dD}{dy} - N\frac{dD}{dz} = 0.$$

Now, since $\dfrac{dN}{dz} = \dfrac{dP}{dy}$, by the conditions of $Mdx + Ndy + Pdz \dots$ being an exact differential, the above equation becomes $P\dfrac{dD}{dy} - N\dfrac{dD}{dz} = 0$, or

$$P\left(\frac{dM}{dy}x + \frac{dN}{dy}y + \frac{dP}{dy}z \dots\right) - N\left(\frac{dM}{dz}x + \frac{dN}{dz}y + \frac{dP}{dz}z \dots\right) = 0.$$

Replacing $\dfrac{dM}{dy}, \dfrac{dP}{dy} \dots$ by $\dfrac{dN}{dx}, \dfrac{dN}{dz} \dots$ and $\dfrac{dM}{dz}, \dfrac{dN}{dz} \dots$ by $\dfrac{dP}{dx}, \dfrac{dP}{dy} \dots$, this equation becomes

$$P\left(\frac{dN}{dx}x + \frac{dN}{dy}y + \frac{dN}{dz}z \dots\right) - N\left(\frac{dP}{dx}x + \frac{dP}{dy}y + \frac{dP}{dz}z \dots\right) = 0.$$

Now $$\frac{dN}{dx}x + \frac{dN}{dy}y + \ldots = nN,$$

$$\frac{dP}{dx}x + \frac{dP}{dy}y + \ldots = nP,$$

and therefore the above equation *is* satisfied. Hence

$$\frac{Ndy + Pdz\ldots}{Mx + Ny + Pz\ldots}, \text{ or its equal } \frac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots},$$

is an exact differential $d\psi(y', z' \ldots)$. Consequently equation (3) becomes

$$dU = (Mx + Ny + Pz \ldots)\, d\left\{\log x + \psi(y', z' \ldots)\right\};$$

which equation being by hypothesis integrable, it follows that

$$Mx + Ny + Pz \ldots = \phi\left\{\log x + \psi(y', z' \ldots)\right\}:$$

and $Mx + Ny \ldots$ being moreover a homogeneous function of $n+1$ dimensions, it is clear that we must have $\phi(u) = A\, e^{(n+1)\alpha}$. Hence we have

$$dU = A\, x^{n+1} e^{n+1)\psi} d(\log x + \psi).$$

If now $n + 1$ is not equal to 0, we have

$$U = \frac{Ax^{n+1} e^{(n+1)\psi}}{n+1},$$

omitting the constant; but if $n = -1$, we have

$$U = A(\log x + \psi) + C.$$

We see then that if $Mx + Ny + Pz \ldots = 0$, which can only happen when $n = -1$, U will be a homogeneous function of $n + 1$ or 0 dimensions. If $Mx + Ny + Pz \ldots\ldots$ is not equal to 0, then, if $n + 1$ is not equal to 0, and the constant in U is properly chosen, U will be a homogeneous function of $n + 1$ dimensions, but if $n + 1 = 0$, U will not be a homogeneous function of 0 dimensions, but will contain $\log x$. Of course it might equally have contained the logarithm of y or z, &c.; in fact,

$$\log x + \psi(y', z' \ldots) = \log y + \log\frac{x}{y} + \psi(y', z' \ldots)$$
$$= \log y + \chi(y', z' \ldots).$$

II.—TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS OF A PLANET'S MOTION.

By B. Bronwin.

Making x, y, z the co-ordinates of a disturbed planet, x', y', z' those of the disturbing body, r and r' the radii-vectores, and

$$R = \frac{m'(xx' + yy' + zz')}{r'^3} - \frac{m'}{\{r'^2 - 2(xx' + yy' + zz') + r^2\}^{\frac{1}{2}}};$$

the known differential equations of its motion are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0, \quad \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0 \quad \ldots . (A).$$

Let i be the inclination of the plane of the orbit to the fixed plane of xy, θ the longitude of the node on the fixed plane, and ϑ its longitude on the plane of the orbit, having a fixed origin on it. Then $np = nn' \cos n'np$ (see fig. 1), or $d\vartheta = \cos i \, d\theta$.

If u and v be the co-ordinates referred to the line of the nodes, $x = u \cos\theta - v \sin\theta$, $y = u \sin\theta + v\cos\theta$. And if w be the co-ordinate on the orbit, of which v is the projection, $v = w\cos i$, $z = w \sin i$. Also, if ξ and η be the co-ordinates on the plane of the orbit, the axis of η passing through the origin of ϑ,

$$u = \xi\cos\vartheta + \eta\sin\vartheta, \quad w = -\xi\sin\vartheta + \eta\cos\vartheta.$$

From these, by eliminating u, v, and w, we have

$$x = A\xi + B\eta, \quad y = C\xi + D\eta, \quad z = E\xi + F\eta \ldots . (1),$$

$A = \cos\theta\cos\vartheta + \cos i \sin\theta\sin\vartheta$, $B = \cos\theta\sin\vartheta - \cos i\sin\theta\cos\vartheta$,
$C = \sin\theta\cos\vartheta - \cos i\cos\theta\sin\vartheta$, $D = \sin\theta\sin\vartheta + \cos i\cos\theta\cos\vartheta$,
$E = -\sin i\sin\vartheta$, $F = \sin i \cos\vartheta$.

Remembering that $d\vartheta = \cos i \, d\theta$, we find

$$dA = -\sin\theta\cos\theta\, d\vartheta - \cos i\cos\theta\sin\vartheta\, d\theta - \sin i\sin\theta\sin\vartheta\, di$$
$$+ \cos i\cos\theta\sin\vartheta\, d\theta + \cos^2 i\sin\theta\cos\vartheta\, d\theta$$
$$= -\sin i \sin\theta\sin\vartheta\, di - \sin^2 i\sin\theta\cos\vartheta\, d\theta;$$

and $dE = -\cos i\sin\vartheta\, di - \sin i\cos i\cos\vartheta\, d\theta$.

From these it will be easily seen that $dA = \tan i \sin\theta\, dE$.

Similar results may be obtained by differentiating the values of B, C, &c. They are, including the one just obtained,

$$\left.\begin{array}{ll} dA = \tan i\sin\theta\, dE, & dB = \tan i \sin\theta\, dF \\ dC = -\tan i\cos\theta\, dE, & dD = -\tan i\cos\theta\, dF \end{array}\right\} \ldots . (2).$$

As we have four quantities to determine and only three equations, we must make an assumption. Assume

$$\xi dE + \eta dF = 0, \ \ \xi dA + \eta dB = 0, \ \ \xi dC + \eta dD = 0 \ldots (3).$$

This is only equivalent to one assumption: for if we put for dA, dB, dC, dD their values from (2), we see that the two last of (3) reduce to the first. The assumption here made is the one usually made. For if $\phi =$ lat. $v =$ long. on the orbit, we have $z = r \sin \phi$, $\xi = r \cos v$, $\eta = r \sin v$; and $z = E\xi + F\eta$ becomes

$$\sin \phi = E \cos v + F \sin v = \sin i (\sin v \cos \vartheta - \cos v \sin \vartheta)$$
$$= \sin i \sin (v - \vartheta);$$

and $\xi dE + \eta dF = 0$ is $\dfrac{d \sin \phi}{di} di + \dfrac{d \sin \phi}{d\vartheta} d\vartheta = 0$.

By virtue of (3) we have

$$\frac{dx}{dt} = A\frac{d\xi}{dt} + B\frac{d\eta}{dt}, \quad \frac{dy}{dt} = C\frac{d\xi}{dt} + D\frac{d\eta}{dt}, \quad \frac{dz}{dt} = E\frac{d\xi}{dt} + F\frac{d\eta}{dt};$$

and therefore

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} + \frac{dA}{dt}\frac{d\xi}{dt} + \frac{dB}{dt}\frac{d\eta}{dt} \\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} + \frac{dC}{dt}\frac{d\xi}{dt} + \frac{dD}{dt}\frac{d\eta}{dt} \\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} + \frac{dE}{dt}\frac{d\xi}{dt} + \frac{dF}{dt}\frac{d\eta}{dt} \end{aligned}\right\} \ldots (4).$$

The following equations of condition will now be necessary:

$$\left.\begin{aligned} &A^2 + C^2 + E^2 = 1, \ B^2 + D^2 + F^2 = 1, \ AB + CD + EF = 0, \\ &AdA + CdC + EdE = 0, \quad BdB + DdD + FdF = 0, \\ &AdB + CdD + EdF = 0, \quad BdA + DdC + FdE = 0, \end{aligned}\right\} \ldots (5).$$

These may be derived as follows. In $r^2 = x^2 + y^2 + z^2 = \xi^2 + \eta^2$ put for x, y, and z their values from (1); the coefficients of ξ^2, $\xi\eta$, η^2, equalled to zero, will give the three first. Differentiation of the first and second will give the fourth and fifth: these last, by means of (3), will give the sixth and seventh. Or if in $xdx + ydy + zdz = \xi d\xi + \eta d\eta$ for x, dx, &c. we substitute their values from (1), the coefficients of the several terms of the result equalled to zero will give them all. And they may be all verified by putting for A, B, &c. and their differentials their values.

We easily derive from (4), attending to (5),

$$\left.\begin{aligned} A\frac{d^2x}{dt^2} + C\frac{d^2y}{dt^2} + E\frac{d^2z}{dt^2} &= \frac{d^2\xi}{dt^2} \\ B\frac{d^2x}{dt^2} + D\frac{d^2y}{dt^2} + F\frac{d^2z}{dt^2} &= \frac{d^2\eta}{dt^2} \end{aligned}\right\} \dots\dots (6),$$

and from (1), having regard to (5),

$$\frac{\mu}{r^3}(Ax + Cy + Ez) = \frac{\mu\xi}{r^3}, \quad \frac{\mu}{r^3}(Bx + Dy + Fz) = \frac{\mu\eta}{r^3} \dots (7).$$

Multiply the first of (A) by A, the second by C, the third by E, and add the results; we have by (6) and (7),

$$\frac{d^2\xi}{dt^2} + \frac{\mu\xi}{r^3} + A\frac{dR}{dx} + C\frac{dR}{dy} + E\frac{dR}{dz} = 0.$$

Again, multiply the first of (A) by B, the second by D, the third by F; add results: we have by (6) and (7),

$$\frac{d^2\eta}{dt^2} + \frac{\mu\eta}{r^3} + B\frac{dR}{dx} + D\frac{dR}{dy} + F\frac{dR}{dz} = 0.$$

But

$$\frac{dR}{d\xi} = \frac{dR}{dx}\frac{dx}{d\xi} + \frac{dR}{dy}\frac{dy}{d\xi} + \frac{dR}{dz}\frac{dz}{d\xi} = A\frac{dR}{dx} + C\frac{dR}{dy} + E\frac{dR}{dz} \text{ by (1).}$$

In like manner $\dfrac{dR}{d\eta} = B\dfrac{dR}{dx} + D\dfrac{dR}{dy} + F\dfrac{dR}{dz}$.

These values of $\dfrac{dR}{d\xi}$, $\dfrac{dR}{d\eta}$ reduce the equations last obtained to

$$\frac{d^2\xi}{dt^2} + \frac{\mu\xi}{r^3} + \frac{dR}{d\xi} = 0, \quad \frac{d^2\eta}{dt^2} + \frac{\mu\eta}{r^3} + \frac{dR}{d\eta} = 0 \dots (B).$$

The equations (B) are the same as they would be if the plane of the orbit were fixed.

Since R is independent of the position of the fixed plane, if we make that plane coincide with the orbit, the quantity $xx' + yy' + zz'$ will become $x'\xi + y'\eta$, or $M\xi + N\eta$. And if we transform that quantity, it will become $M\xi + N\eta$, M and N having the same values in both cases, though functions of different quantities in the two cases. Consequently $\dfrac{dR}{d\xi}$, $\dfrac{dR}{d\eta}$ will have the same values, whatever be the position of the fixed plane.

Make $\dfrac{\xi d\eta - \eta d\xi}{dt} = h$. Then by (3),

$$\frac{dA}{dt}\frac{d\xi}{dt} + \frac{dB}{dt}\frac{d\eta}{dt} = \frac{dA}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\frac{dA}{dt},$$

$$\frac{dC}{dt}\frac{d\xi}{dt} + \frac{dD}{dt}\frac{d\eta}{dt} = \frac{dC}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\frac{dC}{dt},$$

$$\frac{dE}{dt}\frac{d\xi}{dt} + \frac{dF}{dt}\frac{d\eta}{dt} = \frac{dE}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\frac{dE}{dt}.$$

Substituting these values in (4), they become

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\frac{dA}{dt}\\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\frac{dC}{dt}\\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\frac{dE}{dt}\end{aligned}\right\} \quad \ldots. (8).$$

These last may be changed by (3) into

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\frac{dB}{dt}\\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\frac{dD}{dt}\\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\frac{dF}{dt}\end{aligned}\right\} \quad \ldots. (9).$$

By (2) we find

$$\left.\begin{aligned}\left(\frac{dA}{dt}\right)^2 + \left(\frac{dC}{dt}\right)^2 + \left(\frac{dE}{dt}\right)^2 &= \frac{1}{\cos^2 i}\cdot\left(\frac{dE}{dt}\right)^2\\ \left(\frac{dB}{dt}\right)^2 + \left(\frac{dD}{dt}\right)^2 + \left(\frac{dF}{dt}\right)^2 &= \frac{1}{\cos^2 i}\cdot\left(\frac{dF}{dt}\right)^2\end{aligned}\right\} \quad \ldots.(10).$$

We now easily find from (8) and (9), having regard to (5) and (10),

$$\left.\begin{aligned}\frac{dA}{dt}\frac{d^2x}{dt^2} + \frac{dC}{dt}\frac{d^2y}{dt^2} + \frac{dE}{dt}\frac{d^2z}{dt^2} &= -\frac{h}{\eta\cos^2 i}\left(\frac{dE}{dt}\right)^2\\ \frac{dB}{dt}\frac{d^2x}{dt^2} + \frac{dD}{dt}\frac{d^2y}{dt^2} + \frac{dF}{dt}\frac{d^2z}{dt^2} &= \frac{h}{\xi\cos^2 i}\left(\frac{dF}{dt}\right)^2\end{aligned}\right\} \ldots(11).$$

Putting for x, y, and z their values from (1), we find in consequence of (5),

$$\left.\begin{aligned}\frac{\mu}{r^3}\left(\frac{dA}{dt}x + \frac{dC}{dt}y + \frac{dE}{dt}z\right) = 0\\ \frac{\mu}{r^3}\left(\frac{dB}{dt}x + \frac{dD}{dt}y + \frac{dF}{dt}z\right) = 0\end{aligned}\right\} \ldots\ldots (12).$$

Multiply first of (A) by $\frac{dA}{dt}$, second by $\frac{dC}{dt}$, third by $\frac{dE}{dt}$; add results: by (11) and (12) we obtain

$$-\frac{h}{\eta\cos^2 i}\left(\frac{dE}{dt}\right)^2 + \frac{dA}{dt}\,\frac{dR}{dx} + \frac{dC}{dt}\,\frac{dR}{dy} + \frac{dE}{dt}\,\frac{dR}{dz}.$$

Again, multiply first of (A) by $\frac{dB}{dt}$, second by $\frac{dD}{dt}$, third by $\frac{dF}{dt}$; add results: by (11) and (12) we obtain

$$\frac{h}{\xi\cos^2 i}\left(\frac{dF}{dt}\right)^2 + \frac{dB}{dt}\,\frac{dR}{dx} + \frac{dD}{dt}\,\frac{dR}{dy} + \frac{dF}{dt}\,\frac{dR}{dz}.$$

Putting for dA, dB, dC, dD their values from (2), the two last found equations reduce to

$$\left.\begin{aligned}\frac{h}{\cos i}.\frac{dE}{dt} &= \eta\sin i\sin\theta\frac{dR}{dx} - \eta\sin i\cos\theta\frac{dR}{dy} + \eta\cos i\frac{dR}{dz}\\ \frac{h}{\cos i}.\frac{dF}{dt} &= -\xi\sin i\sin\theta\frac{dR}{dx} + \xi\sin i\cos\theta\frac{dR}{dy} - \xi\cos i\frac{dR}{dz}\end{aligned}\right\}$$

Now, by (1) and (2), $\ldots\ldots (13),$

$$\left.\begin{aligned}\frac{dR}{dF} = \frac{dR}{dx}\,\frac{dx}{dF} + \frac{dR}{dy}\,\frac{dy}{dF} + \frac{dR}{dz}\,\frac{dz}{dF} &= \eta\frac{dR}{dx}\,\frac{dB}{dF} + \eta\frac{dR}{dy}\,\frac{dD}{dF} + \eta\frac{dR}{dz}\\ &= \eta\tan i\sin\theta\frac{dR}{dx} - \eta\tan i\cos\theta\frac{dR}{dy} + \eta\frac{dR}{dz}.\\ \text{In like manner}\qquad&\\ \frac{dR}{dE} = \xi\tan i\sin\theta\frac{dR}{dx} - \xi\tan i&\cos\theta\frac{dR}{dy} + \xi\frac{dR}{dz}.\end{aligned}\right\}$$

By means of (14), (13) will become $\ldots\ldots (14).$

$$\frac{h}{\cos^2 i}\,\frac{dE}{dt} = \frac{dR}{dF},\quad \frac{h}{\cos^2 i}\,\frac{dF}{dt} = -\frac{dR}{dE}\ldots\ldots (15).$$

To follow a usual notation, make

$$p = -E = \sin i\sin\vartheta,\quad q = F = \sin i\cos\vartheta,$$

and (15) become

$$dp = -\frac{\cos^2 i\, dt}{h}\frac{dR}{dq},\quad dq = \frac{\cos^2 i\, dt}{h}\frac{dR}{dp} \ldots (16),$$

where $\cos^2 i = 1 - p^2 - q^2$.

From equations (16) the following formulæ, which are sometimes used, may easily be deduced:

$$d\theta = -\frac{dt}{h\sin i}\frac{dR}{di},\quad d\vartheta = -\frac{\cos i\, dt}{h\sin i}\frac{dR}{di},\quad di = \frac{dt}{h\sin i}\frac{dR}{d\theta}.$$

The equation $\sin\phi = E\cos v + F\sin v = \sin i \sin(v - \vartheta)$ will give the latitude by means of E and F, and we may form a differential equation for the purpose if we please. In which case we may find the variations of E and F, or of i and ϑ, due to the several powers of the disturbing force, from those of ϕ and v, in the manner pointed out in my paper on the Differential Equations of the Moon's motion in the *London, Edinburgh, and Dublin Philosophical Magazine*, Feb. 1844. Those equations are founded on the previous knowledge of equations (B) of this paper; which last are convenient for finding the co-ordinates r and v on the plane of the orbit instead of their projected values on the fixed plane. And this method is mostly the easiest way of solving the problem.

III.—ON A NEW METHOD OF OBTAINING THE EXPRESSION FOR THE SINE AND COSINE OF THE MULTIPLE ARC IN TERMS OF THE LIKE FUNCTIONS OF THE SIMPLE ARC.

By R. MOON, M.A. Fellow of Queens' College.

THE subject which I propose to discuss in the following short paper is somewhat trite; but as the method I purpose to adopt possesses great simplicity, and as the artifice adopted in it may be usefully extended to other cases, I perhaps may be excused in my endeavour to call attention to it.

By De Moivre's Theorem,

$$\cos n\theta + \sqrt{(-1)}\sin n\theta = \{\cos\theta + \sqrt{(-1)}\sin\theta\}^n$$

$$= \cos^n\theta + C_1\sqrt{(-1)}\sin\theta\cos^{n-1}\theta - C_2\sin^2\theta\cos^{n-2}\theta$$
$$- C_3\sqrt{(-1)}\sin^3\theta\cos^{n-3}\theta + \&c.$$

where C_1, C_2, C_3 .... are the successive coefficients in the expansion of $(1 + x)^n$;

Hence $\cos n\theta = \cos^n\theta - C_2 \sin^2\theta \cos^{n-2}\theta + C_4 \sin^4\theta \cos^{n-4}\theta - \&c.$

It will be observed, that the coefficient of $\cos^{n-2r}\theta$ in the above expression is a homogeneous function of one dimension of the quantities $C_2, C_3, C_4 \ldots$; and this function will be identical with the coefficient of $\cos^{n-2r}\theta$ in the expression

$$u = \cos^n\theta - A(1-\cos^2\theta)\cos^{n-2}\theta + A^2(1-\cos^2\theta)^2\cos^{n-4}\theta - \&c.$$

if we change A into C_2, A^2 into C_4, A^3 into C_6, &c. But

$$u = \cos^n\theta \left\{1 - A\frac{1-\cos^2\theta}{\cos^2\theta} + A^2\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^2 - A^3\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^3 + \&c.\right\}$$

$$= \cos^n\theta\, f\left\{-\left(A\,\frac{1-\cos^2\theta}{\cos^2\theta}\right)\right\}$$

$$= \cos^n\theta\, f\left\{A - \frac{A}{\cos^2\theta}\right\}$$

$$= \cos^n\theta \left\{\begin{array}{l} fA - \frac{1}{1}\,\frac{A}{\cos^2\theta}\,\frac{dfA}{dA} + \frac{1}{1.2}\left(\frac{A}{\cos^2\theta}\right)^2\frac{d^2fA}{dA^2} \\ \qquad - \frac{1}{1.2.3}\left(\frac{A}{\cos^2\theta}\right)^3\frac{d^3fA}{dA^3} \\ \qquad + \&c. \end{array}\right.$$

and fA is the coefficient of $\cos^n\theta$ in the equivalent for u; therefore

$$fA = 1 + A + A^2 + A^3 + A^4 + \&c.$$

$$\frac{dfA}{dA^2} = 1 + 2A + 3A^2 + 4A^3 + \&c.$$

$$\frac{d^2fA}{dA^2} = 2.1 + 3.2A + 4.3A^2 + \&c.$$

$$\frac{d^3fA}{d\theta^3} = 3.2.1 + 4.3.2A + \&c.$$

$$\ldots\ldots = \ldots\ldots$$

Hence
$$\begin{aligned} u = {} & \cos^n\theta\,(1 + A + A^2 + A^3 + \&c.) \\ & - \frac{\cos^{n-2}\theta}{1}(A + 2A^2 + 3A^3 + 4A^4 + \&c.) \\ & + \frac{\cos^{n-4}\theta}{1.2}(1.2A^2 + 2.3A^3 + 3.4A^4 + \&c.) \\ & - \frac{\cos^{n-6}\theta}{1.2.3}(1.2.3A^3 + 2.3.4A^4 + 3.4.5A^5 + \&c.) + \&c.; \end{aligned}$$

and substituting for A, A^2, A^3 .... &c., we get

$$\begin{aligned}\cos n\theta = {} & \cos^n\theta\,(1 + C_2 + C_4 + C_6 + \&c.) \\ & - \frac{\cos^{n-2}\theta}{1}(C_2 + 2C_4 + 3C_6 + \&c.) \\ & + \frac{\cos^{n-4}\theta}{1.2}(1.2C_4 + 2.3C_6 + 3.4C_8 + \&c.) \\ & - \frac{\cos^{n-6}\theta}{1.2.3}(1.2.3C_6 + 2.3.4C_8 + 3.4.5C_{10} + \&c. \\ & + \&c.\end{aligned}$$

a result which may be readily verified.

Similarly we have

$$\begin{aligned}\sin n\theta & = C_1 \sin\theta\cos^{n-1}\theta - C_3\sin^3\theta\cos^{n-3}\theta + C_5\sin^5\theta\cos^{n-5}\theta - \&c. \\ & = \sin\theta\,\{C_1\cos^{n-1}\theta - C_3(1-\cos^2\theta)\cos^{n-3}\theta \\ & \qquad + C_5(1-\cos^2\theta)^2\cos^{n-5}\theta - \&c. \\ & = \sin\theta\,.\,u.\end{aligned}$$

Now the coefficient of $(\cos\theta)^{n-p}$ in u may be found from that of $(\cos\theta)^{n-p}$ in the expression

$$v = A\cos^{n-1}\theta - A^2(1-\cos^2\theta)\cos^{n-3}\theta + A^3(1-\cos^2\theta)^2\cos^{n-5}\theta - \&c.$$

by substituting C_2 for A, C_3 for A^2, C_5 for A^3, &c.

But $v = A\cos^{n-1}\theta\left\{1 - A\dfrac{1-\cos^2\theta}{\cos^2\theta} + A^2\left(\dfrac{1-\cos^2\theta}{\cos^2\theta}\right)^2 - \&c.\right\}$

$$= A\cos^{n-1}\theta\, f\left(A - \frac{A}{\cos^2\theta}\right);$$

whence we easily find

$$\sin n\theta = \sin\theta \left\{\begin{aligned} & \cos^{n-1}\theta\,(C_1 + C_3 + C_5 + \&c.) \\ & - \frac{\cos^{n-3}\theta}{1}(C_3 + 2C_5 + 3C_7 + \&c.) \\ & + \frac{\cos^{n-5}\theta}{1.2}(1.2C_5 + 2.3C_7 + 3.4C_9 + \&c.) \\ & - \frac{\cos^{n-7}\theta}{1.2.3}(1.2.3C_7 + 2.3.4C_9 + \&c.) + \&c.\end{aligned}\right.$$

We might in the same manner easily obtain the expression for $\sin n\theta$ and $\cos n\theta$ in terms of the powers of $\sin\theta$.

Liverpool, Dec. 20, 1844.

IV.—ON INDETERMINATE MAXIMA AND MINIMA.

By W. Walton, M.A., Trinity College.

Suppose that we have n equations connecting a quantity r with n quantities $x, y, z, \ldots$ and let it be proposed to determine a relation among certain parameters of these equations, such that for all values of $x, y, z, \ldots$ the value of r shall remain invariable. Problems coming under this general head may generally be solved with much elegance by the following method. Proceed by the ordinary rules to find the values of $x, y, z, \ldots$ which correspond to a maximum or minimum value of r: the result of the investigation will be an equation which, together with the n original equations, will generally serve to determine $x, y, z, \ldots$ as well as r. Having obtained these equations, we must assign such relations among the proper parameters of the equations as shall render the values of $x, y, z, \ldots$ all or some of them, according to the case, indeterminate. These relations will constitute the solution of the problem proposed. I will subjoin the solutions of several problems of the above class, which will sufficiently elucidate the principles of the method.

(1) To investigate the positions of the circular sections of the surface

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2 \ldots\ldots (1),$$

made by planes passing through the centre of the surface.

Let (l, m, n) be the direction-cosines of any plane section through the origin, r any radius of the section, and α, β, γ, its direction-cosines. Then

$$l\alpha + m\beta + n\gamma = 0 \ldots\ldots (2),$$

and

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \ldots\ldots (3);$$

also, from (1), we have

$$r^2 = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 \ldots\ldots (4).$$

Since r must be constant for a circular section, differentiating (2), (3), (4), on this hypothesis with respect to α, β, γ, we have

$$l d\alpha + m d\beta + n d\gamma = 0,$$
$$\alpha d\alpha + \beta d\beta + \gamma d\gamma = 0,$$
$$a^2\alpha d\alpha + b^2\beta d\beta + c^2\gamma d\gamma = 0.$$

Multiplying these three equations in order by $1, -\lambda\mu, -\lambda$, and adding, we get, $\lambda\mu$ and λ being arbitrary multipliers,

$$\left.\begin{aligned} l &= \lambda(\mu - a^2)\alpha \\ m &= \lambda(\mu - b^2)\beta \\ n &= \lambda(\mu - c^2)\gamma \end{aligned}\right\} \ldots\ldots (5).$$

From (2) and (5) there is

$$\frac{l^2}{\mu - a^2} + \frac{m^2}{\mu - b^2} + \frac{n^2}{\mu - c^2} = 0 \ldots\ldots (6):$$

from (3) and (5),

$$\frac{l^2}{(\mu - a^2)^2} + \frac{m^2}{(\mu - b^2)^2} + \frac{n^2}{(\mu - c^2)^2} = \lambda^2 \ldots\ldots (7).$$

From (3) and (4),

$$\lambda^2 r^2 = \frac{l^2 a^2}{(\mu - a^2)^2} + \frac{m^2 b^2}{(\mu - b^2)^2} + \frac{n^2 c^2}{(\mu - c^2)^2}$$

$$= \mu\lambda^2, \quad \text{by } \mu(7) - (6),$$

whence $$r^2 = \mu \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now, the section being circular, α, β, γ, must each be an indeterminate quantity: hence, as is evident from (5), the value of λ in (7) must be of the form $\frac{0}{0}$. We must have, therefore, one of the three relations

$$\left\{\begin{matrix} l^2 = 0 \\ r^2 = a^2 \end{matrix}\right\}, \quad \left\{\begin{matrix} m^2 = 0 \\ r^2 = b^2 \end{matrix}\right\}, \quad \left\{\begin{matrix} n^2 = 0 \\ r^2 = c^2 \end{matrix}\right\}.$$

Corresponding to each of these relations, we must have respectively, from (2), (5), (8),

$$\frac{m^2}{a^2 - b^2} + \frac{n^2}{a^2 - c^2} = 0, \quad \frac{n^2}{b^2 - c^2} + \frac{l^2}{b^2 - a^2} = 0, \quad \frac{l^2}{c^2 - a^2} + \frac{m^2}{c^2 - b^2} = 0:$$

of which relations the second alone is possible. Thus we see that there are two circular sections given by the equation

$$\frac{l}{n} = \pm\left(\frac{a^2 - c^2}{b^2 - c^2}\right)^{\frac{1}{2}}, \quad m = 0;$$

the radii of both being equal to b.

If any other possible relation were established between l, m, n, the values of α, β, γ, r, might be found from the equations (5), (6), (7), (8); so that r would be ascertained both in magnitude and position. Thus the problem with which we have been occupied is a nugatory case of a general problem of maximum or minimum values.

(2) To investigate the positions of the circular sections of an ellipsoid made by planes passing through the centre.

The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

whence
$$\frac{1}{r^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2},$$
it is evident that all the equations of the last problem will be adapted to the case of the ellipsoid by writing $\frac{1}{r^2}, \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, respectively in place of r^2, a^2, b^2, c^2.

Thus we see that the positions of the circular sections will be determined by one of the relations
$$\frac{m^2}{\frac{1}{a^2}-\frac{1}{b^2}} + \frac{n^2}{\frac{1}{a^2}-\frac{1}{c^2}} = 0, \quad \frac{n^2}{\frac{1}{b^2}-\frac{1}{c^2}} + \frac{l^2}{\frac{1}{b^2}-\frac{1}{a^2}} = 0, \quad \frac{l^2}{\frac{1}{c^2}-\frac{1}{a^2}} + \frac{m^2}{\frac{1}{c^2}-\frac{1}{b^2}} = 0,$$
the second only of which is possible. Thus we see that two circular sections are determined by the equations
$$m = 0, \quad \frac{l}{n} = \pm \left\{ \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{c^2}-\frac{1}{b^2}} \right\}^{\frac{1}{2}}.$$

(3) To find the position of a plane passing through the centre of an ellipsoid, such that the perpendiculars from the centre upon the tangent planes at every point of the curve of section shall be equal.

The equation to the ellipsoid being
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ \ldots\ldots\ldots \ (1),$$
we shall have, p being one of the perpendiculars,
$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \ \ldots\ldots\ldots \ (2).$$

Let the equation to the plane of section be
$$lx + my + nz = 0 \ \ldots\ldots \ (3).$$
Since p is to be constant, we have from (3), (1), (2),
$$l\,dx + m\,dy + n\,dz = 0,$$
$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz = 0,$$
$$\frac{x}{a^4}\,dx + \frac{y}{b^4}\,dy + \frac{z}{c^4}\,dz = 0.$$

Multiplying these equations by $1, -\lambda\mu, -\lambda$, and adding, we shall get, $\lambda\mu$ and λ being arbitrary,

$$\left.\begin{aligned} l &= \frac{\lambda}{a^2}\left(\mu - \frac{1}{a^2}\right)x, \\ m &= \frac{\lambda}{b^2}\left(\mu - \frac{1}{b^2}\right)y, \\ n &= \frac{\lambda}{c^2}\left(\mu - \frac{1}{c^2}\right)z. \end{aligned}\right\} \ldots\ldots (4).$$

From (1) and (4),

$$\frac{l^2a^2}{\left(\mu - \frac{1}{a^2}\right)^2} + \frac{m^2b^2}{\left(\mu - \frac{1}{b^2}\right)^2} + \frac{n^2c^2}{\left(\mu - \frac{1}{c^2}\right)^2} = \lambda^2 \ldots. (5);$$

from (3) and (4),

$$\frac{l^2a^2}{\mu - \frac{1}{a^2}} + \frac{m^2b^2}{\mu - \frac{1}{b^2}} + \frac{n^2c^2}{\mu - \frac{1}{c^2}} = 0 \quad \ldots\ldots\ldots (6).$$

From (2) and (4),

$$\frac{\lambda^2}{p^2} = \frac{l^2}{\left(\mu - \frac{1}{a^2}\right)^2} + \frac{m^2}{\left(\mu - \frac{1}{b^2}\right)^2} + \frac{n^2}{\left(\mu - \frac{1}{c^2}\right)^2}$$

$$= \mu\lambda^2, \text{ by } \mu\,(5) - (6),$$

whence $$\frac{1}{p^2} = \mu \ldots\ldots\ldots\ldots\ldots (7).$$

Since λ, given by (5), must be indeterminate, in order to render x, y, z, indeterminate in (4), we must have one of the relations

$$\left\{\begin{aligned} l &= 0 \\ \mu &= \frac{1}{a^2} \end{aligned}\right\}, \quad \left\{\begin{aligned} m &= 0 \\ \mu &= \frac{1}{b^2} \end{aligned}\right\}, \quad \left\{\begin{aligned} n &= 0 \\ \mu &= \frac{1}{c^2} \end{aligned}\right\}.$$

Corresponding respectively to these relations we have, from (3), (4), (7),

$$\frac{m^2b^2}{\frac{1}{a^2} - \frac{1}{b^2}} + \frac{n^2c^2}{\frac{1}{a^2} - \frac{1}{c^2}} = 0, \qquad \frac{n^2c^2}{\frac{1}{b^2} - \frac{1}{c^2}} + \frac{l^2a^2}{\frac{1}{b^2} - \frac{1}{a^2}} = 0,$$

$$\frac{l^2a^2}{\frac{1}{c^2} - \frac{1}{a^2}} + \frac{m^2b^2}{\frac{1}{c^2} - \frac{1}{b^2}} = 0,$$

of which the second alone is possible.

Thus we see that there are two sections determined by the equations

$$m = 0, \quad \frac{l}{n} = \pm \frac{c}{a}\left\{\frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{b^2}}\right\}^{\frac{1}{2}},$$

the constant magnitude of the perpendicular being $= b^2$.

V.—ON THE INVERSE ELLIPTIC FUNCTIONS.

By A. CAYLEY, M.A. Fellow of Trinity College.

THE properties of the inverse elliptic functions have been the object of the researches of the two illustrious analysts, Abel and Jacobi. Among their most remarkable ones may be reckoned the formulæ given by Abel (*Œuvres*, t. I. p. 213), in which the functions ϕa, fa, Fa, (corresponding to Jacobi's $\sin am.a$, $\cos am.a$, $\Delta am.a$, though not precisely equivalent to these, Abel's radical being $[(1 - c^2x^2)(1 + e^2x^2)]^{\frac{1}{2}}$, and Jacobi's, like that of Legendre's $[(1 - x^2)(1 - k^2x^2)]^{\frac{1}{2}}$), are expressed in the form of fractions, having a common denominator; and this, together with the three numerators, resolved into a doubly infinite series of factors; *i.e.* the general factor contains two independent integers. These formulæ may conveniently be referred to as "Abel's double factorial expressions" for the functions ϕ, f, F. By dividing each of these products into an infinite number of partial products, and expressing these by means of circular or exponential functions, Abel has obtained (p. 216–218) two other systems of formulæ for the same quantities, which may be referred to as "Abel's first and second single factorial systems." The theory of the functions forming the above numerators and denominator, is mentioned by Abel in a letter to Legendre (*Œuvres*, t. II. p. 259), as a subject to which his attention had been directed, but none of his researches upon them have ever been published. Abel's double factorial expressions have nowhere any thing analogous to them in Jacobi's *Fund. Nova;* but the system of formulæ analogous to the first single factorial system is given by Jacobi (p. 86), and the second system is implicitly contained in some of the subsequent formulæ. The functions forming the numerator and denominator of $\sin am.u$, Jacobi represents, omitting a constant factor, by $H, (u)$, $\Theta(u)$; and proceeds to investigate the properties of these new functions. This he principally effects by means of a very remarkable equation of the form

$$l\Theta(u) = \tfrac{1}{2}Au^2 + B\int_0 du.\int_0 du \sin^2 amu,$$

(*Fund. Nova*, p. 145, 133), by which $\Theta(u)$ is made to depend on the known function sin *am.u.* The other two numerators are easily expressed by means of the two functions H, Θ.

From the omission of Abel's double factorial expressions, which are the only ones which display clearly the real nature of the functions in the numerators and denominators; and besides, from the different form of Jacobi's radical, which complicates the transformation from an impossible to a possible argument, it is difficult to trace the connection between Jacobi's formulæ; and in particular to account for the appearance of an exponential factor which runs through them. It would seem therefore natural to make the whole theory depend upon the definitions of the new transcendental functions to which Abel's double factorial expressions lead one, even if these definitions were not of such a nature, that one only wonders they should never have been assumed *à priori* from the analogy of the circular functions sin. cos. and quite independently of the theory of elliptic integrals. This is accordingly what I have done in the present paper, in which therefore I assume no single property of elliptic functions, but demonstrate them all, from my fundamental equations. For the sake however of comparison, I retain entirely the notation of Abel. Several of the formulæ that will be obtained are new.

The infinite product

$$x\Pi\left(1+\frac{x}{m\omega}\right)\ldots\ldots\ldots\ldots\ldots(1),$$

where m receives the integer values $\pm 1, \pm 2, \ldots \pm r$, converges, as is well known, as r becomes indefinitely great to a determinate function $\sin\dfrac{\pi x}{\omega}$ of x; the theory of which might, if necessary, be investigated from this property assumed as a definition. We are thus naturally led to investigate the properties of the new transcendant

$$u = x\Pi\Pi\left(1+\frac{x}{m\omega+nvi}\right)\ldots\ldots\ldots(2):$$

m and n are integer numbers, positive or negative; and it is supposed that whatever positive value is attributed to either of these, the corresponding negative one is also given to it. $i=\sqrt{(-1)}$, ω and v are real positive quantities. (At least this is the standard case, and the only one we shall explicitly consider. Many of the formulæ obtained are true, with slight modifications, whatever ω and v represent, provided only $\omega : vi$

be not a real quantity; for if it were so, $m\omega + n\upsilon i$ for some values of m, n would vanish, or at least become indefinitely small, and u would cease to be a determinate function of x).*

Now the value of the above expression, or, as for the sake of shortness it may be written, of the function

$$u = x\Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\}\dots\dots\dots\dots(3),$$

depends in a remarkable manner on the mode in which the superior limits of m, n are assigned. Imagine m, n to have any positive or negative integer values satisfying the equation

$$\phi(m^2, n^2) < T.$$

Consider, for greater distinctness, m, n as the co-ordinates of a point; the equation $\phi(m^2, n^2) = T$ belongs to a certain curve symmetrical with respect to the two axes. I suppose besides that this is a continuous curve without multiple points, and such that the minimum value of a radius vector through the origin continually increases as T increases, and becomes infinite with T. The curve may be *analytically* discontinuous, this is of no importance. The condition with respect to the limits is then that m and n must be integer values denoting the co-ordinates of a point *within* the above curve, the whole system of such integer values being successively taken for these quantities.

Suppose, next, u' denotes the same function as u, except that the limiting condition is

$$\phi'(m^2, n^2) < T'\dots\dots\dots\dots(5).$$

The curve $\phi'(m^2, n^2) = T'$ is supposed to possess the same properties with the other limiting curve, and, for greater distinctness, to lie entirely outside of it; but this last condition is nonessential.

These conditions being satisfied, the ratio $u' : u$ is very easily determined in the limiting case of T and T' infinite. In fact

$$\frac{u'}{u} = \Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\}\dots\dots\dots(6),$$

or

$$\frac{lu'}{u} = \Sigma\Sigma l\left\{1 + \frac{x}{(m, n)}\right\}\dots\dots\dots(7),$$

the limiting conditions being

$$\phi(m^2, n^2) > T\dots\dots\dots\dots(8).$$
$$\phi'(m^2, n^2) < T'.$$

* I have examined the case of impossible values of ω and υ in a paper which I am preparing for *Crelles' Journal*.

Now $$l\left\{1+\frac{x}{(m,\,n)}\right\}=\frac{x}{(m,\,n)}-\tfrac{1}{2}.\frac{x^2}{(m,\,n)^2}+\ldots.\ (9),$$

$$\frac{lu'}{u}=x.\Sigma\Sigma\frac{1}{(m,\,n)}-\tfrac{1}{2}x^2.\Sigma\Sigma\frac{1}{(m,\,n)^2}+\ldots.(10).$$

Or, the alternate terms vanishing on account of the positive and negative values destroying each other,

$$l\frac{u'}{u}=-\tfrac{1}{2}x^2.\Sigma\Sigma\frac{1}{(m,\,n)^2}-\tfrac{1}{4}x^4.\Sigma\Sigma\frac{1}{(m,\,n)^4}-\ldots.(11).$$

In general $\Sigma\Sigma\psi(m,\,n)=\int\!\!\int\psi(m,\,n)\,dmdn+P\ldots.(12)$,
P denoting a series the first term of which is of the form $C\psi(m,\,n)$, and the remaining ones depending on the differential coefficients of this quantity with respect to m and n. The limits between which the two sides are to be taken, are identical.

In the present case, supposing T and T' indefinitely great, it is easy to see that the first term of the expression for $l\,\dfrac{u'}{u}$ is the only one which is not indefinitely small; and we have

$$l\frac{u'}{u}=-\tfrac{1}{2}Ax^2,\quad\text{or}\quad u'=u\varepsilon^{-\frac{1}{2}Ax^2}\ldots\ldots.(13),$$

where $$A=\int\!\!\int\frac{dm.dn}{(m,n)^2}=\int\!\!\int\frac{dm.\,dn}{(m\omega+nvi)^2}\ \ldots\ldots.(14);$$

the limits of the integration being given by

$$\phi\,(m^2,\,n^2)>T.\ldots\ldots\ldots\ldots.(15).$$
$$\phi'(m^2,\,n^2)<T'.$$

Some particular cases are important. Suppose the limits of u' are given by $\quad m^2\omega^2<T^2,\ \ n^2v^2<T^2\ \ldots\ldots\ldots.(16).$

And thus of u, by $\quad m^2\omega^2+n^2v^2.<T^2\ \ldots\ldots\ldots\ldots.(17),$

we have $$A=\int\!\!\int\frac{dmdn}{(m\omega+nvi)^2}\ \ldots\ldots\ldots.\ (18),$$

$$=-\frac{1}{\omega}\int dn.\left\{\frac{1}{T+nvi}-\frac{1}{\sqrt{(T^2-n^2v^2)}+nvi}-\frac{1}{-\sqrt{(T^2-n^2v^2)}+nvi}\right.$$
$$\left.-\frac{1}{-T+nvi}\right\}$$

$$=-\frac{2}{\omega}\int dn.\left\{\frac{T}{T^2+n^2v^2}-\frac{\sqrt{(T^2-n^2v^2)}}{T^2}\right\}\quad(nv=-T,\ nv=T)$$

$$=-\frac{2}{\omega v}\int_{-1}^{1}d\theta.\left\{\frac{1}{1+\theta^2}-\sqrt{(1-\theta^2.)}\right\}=-\frac{2}{\omega v}(\pi-\pi)=0.$$

Or, in this case, $u' = u \ldots\ldots\ldots\ldots\ldots (19).$

Again, let the limits of u' be

$$m^2\omega^2 < R'^2, \quad n^2v^2 < S'^2 \ldots\ldots\ldots\ldots (20),$$

and those of u, $m^2\omega^2 < R^2, \quad n^2v^2 < S^2 \ldots\ldots\ldots\ldots (21).$

$$A = \iint \frac{dmdn}{(m\omega + nvi)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

$$= -\frac{1}{\omega}\int dn.\left\{\frac{1}{R' + nvi} - \frac{1}{R + nvi} + \frac{1}{-R + nvi} - \frac{1}{-R' + nvi}\right\},$$

where the limits are $n^2v^2 < S'^2$, for the terms containing R', $n^2v^2 < S^2$. for the terms containing R

$$= -\frac{2}{\omega vi}.\, l\, \frac{R' + S'i}{R' - S'i}\,\frac{R - Si}{R + Si} \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

$$= -\frac{4}{\omega v}(\lambda' - \lambda), \quad \text{if } \lambda' = \tan^{-1}\frac{S'}{R'}, \; \lambda = \tan^{-1}\frac{S}{R},$$

the arcs λ, λ' being included between the limits $0, \frac{\pi}{2}$. Hence

$$u' = \varepsilon^{2(\lambda' - \lambda)\frac{x^2}{\omega v}}\, u \ldots\ldots\ldots\ldots (24).$$

In particular if $\frac{S'}{R'} = \frac{S}{R}$, $u' = u$. If $\frac{S'}{R'} = 0$, $\frac{S}{R} = 1$, $u' = u\varepsilon^{-\frac{1}{2}\beta x^2}$ if $\frac{S'}{R'} = \infty$, $\frac{S}{R} = 1$, $u' = u\varepsilon^{\frac{1}{2}\beta x^2}$ where $\beta = \frac{\pi}{\omega v}$, for which quantity it will continue to be used.

We may now completely define the functions whose properties are to be investigated. Writing, for shortness,

$$(m, n) = m\omega + nvi \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

$$(\overline{m}, n) = (m + \tfrac{1}{2})\,\omega + nvi.$$

$$(m, \overline{n}) = m\omega + (n + \tfrac{1}{2})\,vi.$$

$$(\overline{m}, \overline{n}) = (m + \tfrac{1}{2})\,\omega + (n + \tfrac{1}{2})\,vi.$$

We may put $\gamma x = x\Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\} \ldots\ldots\ldots\ldots (B).$

$$gx = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, n)}\right\}.$$

$$Gx = \Pi\Pi\left\{1 + \frac{x}{(m, \overline{n})}\right\}.$$

$$\mathfrak{G}x = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, \overline{n})}\right\}.$$

The limits being given respectively by the equations

$$\bmod^s(m, n) < T,\ \bmod^s(\overline{m}, n) < T,\ \bmod^s(m, \overline{n}) < T,\ \bmod^s(\overline{m}, \overline{n}) < T,$$

T being finally infinite. The system of values $m = 0$, $n = 0$, is of course omitted in γx.

The functions γx, gx, Gx, $\mathfrak{G}x$, are all of them real finite functions of x, possessing properties analogous to that of (u). Thus, representing any one of them by Jx, we have

$$Jx = \varepsilon^{\pm\frac{1}{2}\beta x^2} J_{\pm\beta} x \ldots\ldots\ldots\ldots\ldots (C),$$

where $J_{\pm\beta}x$ is the same as Jx, only for $J_\beta x$ the limits are given by $m^2\omega^2$ or $(m+\frac{1}{2})^2\omega^2 < R^2$, n^2v^2 or $(n+\frac{1}{2})^2v^2 < S$, (R, S, and $\frac{R}{S}$ infinite), and for $J_{-\beta}x$, by the same formulæ, (R, S, and $\frac{S}{R}$ infinite). It is to this equation that the most characteristic properties of the functions Jx are due.

The following equations are deduced immediately from the above definitions:

$$\gamma(-x) = -\gamma x,\ g(-x) = gx,\ G(-x) = Gx,\ \mathfrak{G}(-x) = \mathfrak{G}x \ldots (D),$$

$$\gamma(0) = 0,\ g(0) = 1,\ G(0) = 1,\ \mathfrak{G}0 = 1 \ldots\ldots\ldots\ldots\ldots (E),$$

$$\gamma' 0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (F).$$

Suppose $\gamma_1 x$, $g_1 x$, $G_1 x$, $\mathfrak{G}_1 x$, are the values that would have been obtained for γx, gx, Gx, $\mathfrak{G}x$; by interchanging ω and v,—then changing x into xi, and interchanging m and n, by which means the limiting equations are the same in the two cases, we obtain the following system of equations:

$$\gamma_1(xi) = i\gamma x \ldots\ldots\ldots\ldots\ldots (G).$$
$$g_1(xi) = Gx.$$
$$G_1(xi) = gx.$$
$$\mathfrak{G}_1(xi) = \mathfrak{G}x.$$

Or otherwise,

$$\gamma(xi) = i\gamma_1 x \ldots\ldots\ldots\ldots\ldots (H).$$
$$g(xi) = G_1 x.$$
$$G(xi) = g_1 x.$$
$$\mathfrak{G}(xi) = \mathfrak{G}_1 x,$$

equations which are useful in transforming almost any other property of the functions J.

The functions $J^\beta x$ are changed one into another, except as regards a constant multiplier, by the change of x into $x + \frac{\omega}{2}$.

This will be shown in a note, or it may be seen from some formulæ deduced immediately from the definitions of the functions $J_\beta x$, which will be given in the sequel.* Observing the relation between Jx and $J_\beta x$, we have in particular

$$\gamma\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{4}\beta\omega x}\,Agx \ldots\ldots\ldots\ldots (G).$$

$$g\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{4}\omega\beta x}\,B\gamma x.$$

$$G\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{4}\beta\omega x}\,C\mathfrak{G}x.$$

$$\mathfrak{G}\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{4}\beta\omega x}\,DGx,$$

where A, B, C, D, are most simply determined by writing $x=0$, $x=-\frac{\omega}{2}$. Putting at the same time $\varepsilon^{\beta\omega^2}=\varepsilon^{\frac{\pi\omega}{\upsilon}}=q_1^{-1}$,

$$A=\gamma\,\frac{\omega}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (H).$$

$$B=-\,q_1^{-\frac{1}{4}}\div\gamma\left(\frac{\omega}{2}\right).$$

$$C=G\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div\mathfrak{G}\,\frac{\omega}{2}.$$

$$D=\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div G\,\frac{\omega}{2};$$

whence also

$$G\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\ldots\ldots\ldots\ldots (25).$$

Similarly, the functions $J_{-\beta}x$ are changed one into the other by the change of x into $x+\frac{1}{2}\,\upsilon i$. We have in the same way

$$\gamma\left(x+\frac{\upsilon i}{2}\right)=\varepsilon^{-\frac{1}{4}\beta\upsilon x i}\,A'Gx\ldots\ldots\ldots (I).$$

$$g\left(x+\frac{\upsilon i}{2}\right)=\varepsilon^{-\frac{1}{4}\beta\upsilon x i}\,B'\mathfrak{G}x.$$

$$G\left(x+\frac{\upsilon i}{2}\right)=\varepsilon^{-\frac{1}{4}\beta\upsilon x i}\,C'\gamma x.$$

$$\mathfrak{G}\left(x+\frac{\upsilon i}{2}\right)=\varepsilon^{-\frac{1}{4}\beta\upsilon x i}\,D'gx.$$

* Not given in the present paper.

Whence $$A' = \gamma\,\frac{vi}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ (J).$$

$$B' = g\,\frac{vi}{2} = q^{-\frac{1}{4}} \div \mathfrak{G}\,\frac{vi}{2},$$

$$C' = -\,q^{-\frac{1}{4}} \div \gamma\,\frac{vi}{2},$$

$$D' = \mathfrak{G}\,\frac{vi}{2} = q^{-\frac{1}{4}} \div g\,\frac{vi}{2};$$

where $\varepsilon^{\beta v^2} = \varepsilon^{\frac{\pi v}{\omega}} = q^{-1}$. It is obvious that the relation between q and q_1 is $lq\,.\,lq_1 = -\,\pi^2$.

We obtain from the above

$$g\,\frac{vi}{2}\,\mathfrak{G}\,\frac{vi}{2} = q^{-\frac{1}{4}}\ \ldots\ldots\ldots\ (26).$$

Also, by making $x = \frac{vi}{2}$ in the expression for $\gamma\left(x + \frac{\omega}{2}\right)$ and $x = \frac{\omega}{2}$ in that for $\gamma\left(x + \frac{vi}{2}\right)$, we have

$$\gamma\left(\frac{\omega}{2}\right) g\left(\frac{vi}{2}\right) = -\,i\,\gamma\,\frac{vi}{2}\,G\,\frac{\omega}{2}\ \ldots.\ (27),$$

and the same or an equivalent one would have been obtained from the functions g, G, $\mathfrak{G}$.

By combining the above systems, we deduce one of the form

$$\gamma\left(x + \frac{\omega}{2} + \frac{vi}{2}\right) = \varepsilon^{\frac{1}{2}\beta x\,(\omega - vi)}\,A''\,\mathfrak{G}x\ \ldots.\ (K),$$

$$g\left(x + \frac{\omega}{2} + \frac{vi}{2}\right) = \varepsilon^{\frac{1}{2}\beta x\,(\omega - vi)}\,B''\,Gx,$$

$$G\left(x + \frac{\omega}{2} + \frac{vi}{2}\right) = \varepsilon^{\frac{1}{2}\beta x\,(\omega - vi)}\,C''\,gx,$$

$$\mathfrak{G}\left(x + \frac{\omega}{2} + \frac{vi}{2}\right) = \varepsilon^{\frac{1}{2}\beta x\,(\omega - vi)}\,D''\,\gamma x.$$

And observing the equation $\varepsilon^{\beta\omega vi} = \varepsilon^{\pi i} = (-\,1)$, with the following values for the coefficients,

$$A'' = (-\,1)^{\frac{1}{4}}\,.\,\gamma\,\frac{\omega}{2}\,g\,\frac{vi}{2}\ \ldots\ldots\ldots\ (L),$$

$$B'' = -\,(-\,1)^{\frac{1}{4}}.\,q_1^{-\frac{1}{4}}\,.\,\gamma\,\frac{vi}{2} \div \gamma\,\frac{\omega}{2},$$

$$C'' = (-1)^{\frac{1}{4}} \cdot \mathfrak{G}\frac{\omega}{2}\,\mathfrak{G}\frac{\upsilon i}{2},$$

$$D'' = -(-1)^{\frac{1}{4}} \cdot q^{-\frac{1}{4}} \cdot \mathfrak{G}\frac{\omega}{2} \div \gamma\frac{\upsilon i}{2}.$$

Collecting the formulæ with connect $\gamma\left(\frac{\omega}{2}\right) \cdot \gamma\left(\frac{\upsilon i}{2}\right) \ldots$ these are

$$g\frac{\omega}{2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (M).$$

$$G\frac{\upsilon i}{2} = 0,$$

$$G\frac{\omega}{2}\,\mathfrak{G}\frac{\omega}{2} = q_1^{-\frac{1}{4}},$$

$$g\frac{\upsilon i}{2}\,\mathfrak{G}\frac{\upsilon i}{2} = q^{-\frac{1}{4}},$$

$$\gamma\frac{\omega}{2} \cdot g\frac{\upsilon i}{2} = -i\gamma\left(\frac{\upsilon i}{2}\right) \cdot G\left(\frac{\omega}{2}\right).$$

And by the assistance of these

$$B''C'' \div A''D'' = B'D' \div A'C' = CD \div AB = -1 \ldots\ldots (28),$$

$$A'B' \div C'D' = -A''B'' \div C''D'' = -\gamma^2\left(\frac{\upsilon i}{2}\right) \div \mathfrak{G}^2\left(\frac{\upsilon i}{2}\right),$$

$$A''C'' \div B''D'' = -A'C' \div B'D' = \gamma^2\left(\frac{\omega}{2}\right) \div \mathfrak{G}^2\left(\frac{\omega}{2}\right),$$

$$AD \div BC = -A'D' \div B'C' = \gamma^2\left(\frac{\omega}{2}\right) \div G^2\left(\frac{\omega}{2}\right) = -\gamma^2\left(\frac{\upsilon i}{2}\right) \div g^2\left(\frac{\upsilon i}{2}\right),$$

which will be required presently.

It is now easy to proceed to the general systems of formulæ,

$$\Theta = (-1)^{mn} \cdot \varepsilon^{\beta x \cdot (m\omega - n\upsilon i)} \cdot q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2} \ldots (M).$$

$$\gamma\{x + (m, n)\} = (-1)^{m+n} \cdot \Theta\gamma x,$$

$$g\{x + (m, n)\} = (-1)^{m} \cdot \Theta g x,$$

$$G\{x + (m, n)\} = (-1)^{n} \cdot \Theta G x,$$

$$\mathfrak{G}\{x + (m, n)\} = \Theta\mathfrak{G}x.$$

$$\Phi = (-1)^{n(m+\frac{1}{2})} \cdot \varepsilon^{\beta x[(m+\frac{1}{2})\omega - n\upsilon i]}\, q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2}.$$

$$\gamma\{x + (\overline{m}, n)\} = (-1)^{m+n} \cdot \Phi A g x,$$

$$g\{x + (\overline{m}, n)\} = (-1)^{m} \cdot \Phi B \gamma x,$$

$$G\{x + (\overline{m}, n)\} = (-1)^{n} \cdot \Phi C \mathfrak{G} x,$$

$$\mathfrak{G}\{x + (\overline{m}, n)\} = \Phi D G x.$$

$$\Psi = (-1)^{m(n+\frac{1}{2})} \cdot \varepsilon^{\beta x.[m\omega - (n+\frac{1}{2})\upsilon i]} q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\gamma\{x + (m, \bar{n})\} = (-1)^{m+n} \cdot \Psi A' G x,$$

$$g\{x + (m, \bar{n})\} = (-1)^m \cdot \Psi B' \mathfrak{G} x,$$

$$G\{x + (m, \bar{n})\} = (-1)^n \cdot \Psi C' \gamma x,$$

$$\mathfrak{G}\{x + (m, \bar{n})\} = \Psi D' g x.$$

$$\Omega = (-1)^{mn + \frac{m}{2} + \frac{n}{2}} \cdot \varepsilon^{\beta x[(m+\frac{1}{2})\omega - (n+\frac{1}{2})\upsilon i]} \cdot q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\gamma\{x + (\bar{m}, \bar{n})\} = (-1)^{m+n} \cdot \Omega A'' \mathfrak{G} x,$$

$$g\{x + (\bar{m}, \bar{n})\} = (-1)^m \cdot \Omega B'' G x,$$

$$G\{x + (\bar{m}, \bar{n})\} = (-1)^n \cdot \Omega C'' g x,$$

$$\mathfrak{G}\{x + (\bar{m}, \bar{n})\} = \cdot \Omega D'' \gamma x.$$

Suppose $x = 0$, we have the new systems,

$$\Theta_0 = (-1)^{mn} \cdot q_1^{-\frac{1}{2}m^2} q^{-\frac{1}{2}n^2} \quad \ldots\ldots \quad (M\,bis).$$

$\gamma(m, n) = 0,$ $\qquad \gamma'(m, n) = (-1)^{m+n}\,\Theta_0,$

$g(m, n) = (-1)^m \cdot \Theta_0,$

$G(m, n) = (-1)^n \cdot \Theta_0,$

$\mathfrak{G}(m, n) = \Theta_0.$

$$\Phi_0 = (-1)^{n \cdot (m+\frac{1}{2})} q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} q^{-\frac{1}{2}n^2}.$$

$\gamma(\bar{m}, n) = (-1)^{m+n} \cdot \Phi_0 \cdot A,$

$g(\bar{m}, n) = 0,$ $\qquad g'(m, n) = (-1)^m \cdot \Phi_0 B,$

$G(\bar{m}, n) = (-1)^n \cdot \Phi_0 C,$

$\mathfrak{G}(\bar{m}, n) = \Phi_0 D.$

$$\Psi_0 = (-1)^{m \cdot (n+\frac{1}{2})} q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$\gamma(m, \bar{n}) = (-1)^{m+n} \cdot \Psi_0 A',$

$g(m, \bar{n}) = (-1)^m \cdot \Psi_0 B',$

$G(m, \bar{n}) = 0,$ $\qquad G'(m, \bar{n}) = (-1)^n \cdot \Psi_0 C',$

$\mathfrak{G}(m, \bar{n}) = \Psi_0 D'.$

$$\Omega_0 = (-1)^{mn + \frac{1}{2}m + \frac{1}{2}n} q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$\gamma(\bar{m}, \bar{n}) = (-1)^{m+n} \cdot \Omega_0 A'',$

$g(\bar{m}, \bar{n}) = (-1)^m \cdot \Omega_0 B'',$

$G(\bar{m}, \bar{n}) = (-1)^n \cdot \Omega_0 C'',$

$\mathfrak{G}(\bar{m}, \bar{n}) = 0,$ $\qquad \mathfrak{G}'(m, \bar{n}) = \Omega_0 D''.$

We obtain immediately, by taking the logarithmic differentials of the functions γx, gx, Gx, $\mathfrak{G}x$, the equations

$$\begin{aligned} \gamma' x \div \gamma x &= \Sigma\Sigma\{x-(m, n)\}^{-1}, \quad m=0,\ n=0 \text{ admissible}, \\ g'x \div gx &= \Sigma\Sigma\{x-(\overline{m}, n)\}^{-1}, \ldots\ldots\ldots\ldots (N), \\ G'x \div Gx &= \Sigma\Sigma\{x-(m, \overline{n})\}^{-1}, \\ \mathfrak{G}'x \div \mathfrak{G}x &= \Sigma\Sigma\{x-(\overline{m}, \overline{n})\}^{-1}, \end{aligned}$$

the limits being the same as in the case of the factorial expressions.

Consider an equation

$$gx.Gx \div \gamma x.\mathfrak{G}x = \Sigma\Sigma\,[\mathfrak{A}\{x-(m, n)\}^{-1} + \mathfrak{B}\{x-(\overline{m}, \overline{n})\}^{-1}] \ldots (29),$$

we have

$$\mathfrak{A} = g(m, n)G(m, n) \div \gamma'(m, n)\,\mathfrak{G}(m, n) = 1 \ldots\ldots\ldots\ldots\ldots (30),$$

$$\mathfrak{B} = g(\overline{m}, \overline{n})G(\overline{m}, \overline{n}) \div \gamma(\overline{m}, \overline{n})\,\mathfrak{G}'(\overline{m}, \overline{n}) = B''C'' \div A''D'' = -1 \ldots (31).$$

(The application of the ordinary method of decomposition into partial fractions, which is in general exceedingly precarious when applied to transcendental functions, is justified here by a theorem of Cauchy's, which will presently be quoted.) We have thus

$$gx\,Gx \div \gamma x\,\mathfrak{G}x = (\gamma' x \div \gamma x) - (\mathfrak{G}'x \div \mathfrak{G}x),$$

and similarly

$$\begin{aligned} gx\,\mathfrak{G}x \div \gamma x\,Gx &= (\gamma' x \div \gamma x) - (G'x \div Gx), \ldots\ (O), \\ Gx\,\mathfrak{G}x \div \gamma x\,gx &= (\gamma' x \div \gamma x) - (g'x \div gx), \\ -\,b^2\gamma x\,\mathfrak{G}x \div gx\,Gx &= (g'x \div gx) - (G'x \div Gx), \\ e^2\gamma x\,gx \div Gx\,\mathfrak{G}x &= (G'x \div Gx) - (\mathfrak{G}'x \div \mathfrak{G}x), \\ c^2\gamma x\,Gx \div \mathfrak{G}x\,gx &= (\mathfrak{G}'x \div \mathfrak{G}x) - (g'x \div gx); \end{aligned}$$

in which we have written

$$\gamma\,\frac{\upsilon i}{2} \div \mathfrak{G}\left(\frac{\upsilon i}{2}\right) = \frac{i}{e} \ldots\ldots\ldots\ldots (32),$$

$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c},$$

$$\gamma\left(\frac{\omega}{2}\right) \div G\left(\frac{\omega}{2}\right) = -i\left(\gamma\,\frac{\upsilon i}{2} \div g\,\frac{\upsilon i}{2}\right) = \frac{1}{b}.$$

Eliminating the derived coefficients,

$$\begin{aligned} G^2x - \mathfrak{G}^2x &= e^2\gamma^2 x, \ldots\ldots\ldots (33), \\ g^2x - G^2x &= -\,b^2\gamma^2 x, \\ \mathfrak{G}^2x - g^2x &= c^2\gamma^2 x. \end{aligned}$$

Adding $b^2 = e^2 + c^2$ or $b = \sqrt{(e^2 + c^2)}$, in which sense it will continue to be used.

Also, $$g^2x = \mathfrak{G}^2x - c^2\gamma^2x, \ldots\ldots\ldots\ldots (P),$$
$$G^2x = \mathfrak{G}^2x + e^2\gamma^2x.$$

Suppose $$\phi x = \gamma x \div \mathfrak{G}x, \ldots\ldots\ldots\ldots (Q).$$
$$fx = gx \div \mathfrak{G}x,$$
$$Fx = Gx \div \mathfrak{G}x.$$

Then $$f^2x = 1 - c^2\phi^2x, \ldots\ldots\ldots\ldots (R),$$
$$F^2x = 1 + e^2\phi^2x,$$

and also $$\phi'x = \quad fx\, Fx, \ldots\ldots\ldots\ldots (S),$$
$$f'x = -\, c^2\phi x\, Fx,$$
$$F'x = \quad e^2\phi x\, fx.$$

Whence, putting for fx, Fx, their values

$$1 = \frac{\phi'x}{\sqrt{(1 - c^2\phi^2x)(1 + e^2\phi^2x)}} \ldots\ldots (T);$$

or writing $\phi x = y$, and integrating,

$$x = \int_0^{\phi x} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}} \ldots\ldots (U),$$

or $$\phi^{-1}y = \int_0^{y} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}},$$

which shows that ϕ is an inverse elliptic function.

The equations which are the foundation of the theory of the functions ϕ, f, F, are deduced immediately from the equations (S). (*Abel Œuvres*, tom. I. p. 143.) These are

$$\phi(x + y) = \frac{\phi x\, fy\, Fy + \phi y\, fx\, Fx}{1 + e^2c^2\,\phi^2x\,\phi^2y} \ldots\ldots (V),$$

$$f(x + y) = \frac{fx\, fy - c^2\,\phi x\,\phi y\, Fx\, Fy}{1 + e^2c^2\,\phi^2x\,\phi^2y},$$

$$F(x + y) = \frac{Fx\, Fy + e^2\,\phi x\,\phi y\, fx\, fy}{1 + e^2c^2\,\phi^2x\,\phi^2y};$$

so that from this point we may take for granted any properties of these functions. We see, for instance, immediately,

$\phi\,\frac{\upsilon i}{2} = \frac{i}{e}$, $\phi\left(\frac{\omega}{2}\right) = \frac{1}{c}$; whence

$$\frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}}, \ldots\ldots (W),$$

$$\frac{\upsilon i}{2} = \int_0^{\frac{i}{e}} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}}, \text{ or } \frac{\upsilon}{2} = \int_0^{\frac{1}{e}} \frac{dy}{\sqrt{(1 + c^2y^2)(1 - e^2y^2)}} \quad \ldots . \text{(X)},$$

which give the values of ω, υ in terms of c, e; values which may be developed in a variety of ways, in infinite series. We may also express $\gamma \dfrac{\omega}{2}$, &c., and consequently A, B...&c., by means of the quantities c, e. We have only to combine the equations

$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c}, \quad \gamma\left(\frac{\upsilon i}{2}\right) \div \mathfrak{G}\frac{\upsilon i}{2} = \frac{i}{e},$$

$$G\frac{\omega}{2} \div \mathfrak{G}\frac{\omega}{2} = \frac{b}{c}, \quad g\frac{\upsilon i}{2} \div \mathfrak{G}\frac{\upsilon i}{2} = \frac{b}{e} \ldots . (34),$$

with the former relations between these quantities, and we have

$$\gamma\frac{\omega}{2} = b^{-\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \gamma\frac{\upsilon i}{2} = ib^{-\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \ldots . (Y),$$

$$g\frac{\omega}{2} = 0, \qquad g\frac{\upsilon i}{2} = b^{\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}.$$

$$G\frac{\omega}{2} = b^{\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \mathfrak{G}\frac{\upsilon i}{2} = 0,$$

$$\mathfrak{G}\frac{\omega}{2} = b^{-\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \mathfrak{G}\frac{\upsilon i}{2} = b^{-\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}.$$

$$A = b^{-\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad A' = ib^{-\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \quad A'' = (-1)^{\frac{1}{4}} c^{-\frac{1}{2}} e^{-\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}}, \quad \ldots . (Z).$$

$$B = -b^{\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad B' = b^{\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \quad B'' = -(-1)^{\frac{1}{4}} ic^{\frac{1}{2}} e^{-\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

$$C = b^{\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad C' = -ib^{\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}, \quad C'' = (-1)^{\frac{1}{4}} c^{-\frac{1}{2}} e^{\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

$$D = b^{-\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad D' = b^{-\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}, \quad D'' = -(-1)^{\frac{1}{4}} ic^{\frac{1}{2}} e^{\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

which are to be substituted in any formulæ into which these quantities enter.

The following is Cauchy's Theorem, (*Exercises de Math.* tom. II. p. 289).

" If in attributing to the modulus r of the variable

$$z = r\{\cos p + \sqrt{(-1)}\sin p\} \ldots\ldots (35),$$

infinitely great values, these can be chosen so that the two functions

$$\frac{fz + f(-z)}{2}, \quad \frac{fz - f(-z)}{2z}, \ldots \ldots (36),$$

sensibly vanish, whatever be the value of p, or vanish in general, though ceasing to do so, and obtaining *finite* values for certain particular values of p; then

$$fx = \mathcal{E} \frac{\{(fz)\}}{x - z} \ldots \ldots \ldots \ldots (37),$$

the integral residue being reduced to its principal value."

To understand this, it is only necessary to remark that the integral residue in question is the series of fractions that would be obtained by the ordinary process of decomposition; and by the principal value is meant, that *all* those roots are to be taken, the modulus of which is not greater than a certain limit, this limit being afterwards made infinite.

Suppose now fx is a fraction, the numerator and denominator of which are monomials of the form $(\gamma x)^l (gx)^m \ldots$, $l, m \ldots$ being positive integers, and of course no common factor being left in the numerator and denominator.

Let λ be the excess of the degree of the denominator over that of the numerator. Suppose the modulus r of (z) has any value not the same with any of the moduli of

$$(m, n), \quad (\overline{m}, n), \quad (m, \overline{n}), \quad (\overline{m}, \overline{n}) \ldots (38).$$

Then we have

$$r(\cos p + i \sin p) = m\omega + nvi + \theta \ldots (39),$$

θ being a finite quantity, such that none of the functions $J\theta$ vanish. m and n are the greatest integer values which allow the possible part of θ and the coefficient of its impossible part to remain positive. We have therefore

$$m^2\omega^2 + n^2v^2 = r^2 - M \ldots \ldots \ldots (40),$$

M being finite; or when r is infinite, at least one of the values m, n is infinite. The function fz reduces itself to the form

$$q_1^{\frac{\lambda m^2}{2}} q^{\frac{\lambda n^2}{2}} \varepsilon^{mA + nB} F \ldots \ldots \ldots (41),$$

where F is finite. Hence q_1 and q being always less than unity, fz, and consequently both $\frac{1}{2}\{fz + f(-z)\}$ and $\frac{1}{2z}\{fz - f(-z)\}$ vanish for $r = \infty$, as long as λ is positive.

In the case of $\lambda = 0$, the conditions are still satisfied, if we suppose fx to denote an uneven function of x: for when $\lambda=0$, the index of exponential in the above expression vanishes,

or fz is constantly finite. But fz being an odd function of z, $fz + f(-z) = 0$. And $\dfrac{1}{2z}\{fz - f(-z)\}$ vanishes for z infinite, on account of the z in the denominator: hence the expansion is admissible in this case. But it is certainly so also, in a great many cases at least, where fz is an even function of z; for these may be deduced from the others by a simple change in the value of the variable. For instance, from the expansion of $\gamma x \div gx$, which is an odd function, by writing $x + \dfrac{\upsilon i}{2}$ for x, we obtain that of $Gx \div \mathfrak{G}x$, which is even.

A case of some importance is when the function is of the above form, multiplied by an exponential $\varepsilon^{\frac{1}{2}ax^2+bx}$. Here writing $z = m\omega + n\upsilon i + \theta$, the admissibility of the formula depends on the evanescence of

$$\varepsilon^{\frac{1}{2}a(m\omega+n\upsilon i)^2}\, q_1^{\frac{1}{2}\lambda m^2}\, q^{\frac{1}{2}\lambda n^2} \;\ldots\ldots (42);$$

or, if $a = h + ki$, this becomes, omitting a finite factor,

$$\varepsilon^{-\frac{1}{2}m^2(\lambda\beta-h)\omega^2 - \frac{1}{2}n^2(\lambda\beta+h)\upsilon^2 - kmn\omega\upsilon} \;\ldots (43),$$

which vanishes if $h^2 + k^2 < \lambda^2\beta^2$, i.e. the modulus of (a) is less than $\lambda\beta$. The limiting case is admissible when the series is convergent.

We obtain in this way a very great variety of formulæ. For instance,

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div \gamma x$$
$$= \Sigma\Sigma\left[(-1)^{-mn-m-n}\,\varepsilon^{\frac{1}{2}a(m,\,n)^2+b(m,\,n)}\,q_1^{\frac{1}{2}m^2}\,q^{\frac{1}{2}n^2}\,\{x-(m,n)\}^{-1}\right] \ldots (A'),$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div gx$$
$$= -b^{\frac{1}{2}}c^{-\frac{1}{2}}\,\Sigma\Sigma\left[(-1)^{-mn-m-\frac{1}{2}n}\,\varepsilon^{\frac{1}{2}a(\bar{m},\,n)^2+b(\bar{m},\,n)}\,q_1^{\frac{1}{2}(m+\frac{1}{2})^2}\,q^{\frac{1}{2}n^2}\,\{x-(\bar{m},n)\}^{-1}\right],$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div Gx$$
$$= ib^{-\frac{1}{2}}e^{-\frac{1}{2}}\,\Sigma\Sigma\left[(-1)^{-mn-\frac{1}{2}m-n}\,\varepsilon^{\frac{1}{2}a(m,\,n)^2+b(m,\,\bar{n})}\,q_1^{\frac{1}{2}m^2}\,q^{\frac{1}{2}(n+\frac{1}{2})^2}\,\{x-(m,\bar{n})\}^{-1}\right],$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div \mathfrak{G}x$$
$$= ic^{-\frac{1}{2}}e^{-\frac{1}{2}}\,\Sigma\Sigma\left[(-1)^{-(m+\frac{1}{2})(n+\frac{1}{2})}\,\varepsilon^{\frac{1}{2}a(\bar{m},\,\bar{n})^2+b(\bar{m},\,\bar{n})}\,q_1^{\frac{1}{2}(m+\frac{1}{2})^2}\,q^{\frac{1}{2}(n+\frac{1}{2})^2}\{x-(\bar{m},\bar{n})\}^{-1}\right],$$

in which the modulus of a must not exceed β: in the limiting cases, for $a = \beta$, b must be entirely impossible, and for $a = -\beta$, b must be entirely real. The formulæ for γx are

$$\varepsilon^{\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma(-1)^{-m-n}\,q^{n^2}\,\varepsilon^{b(m,\,n)}\,\{x-(m,n)\}^{-1} \ldots (44),$$
$$\varepsilon^{-\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma(-1)^{m-n}\,q_1^{m^2}\,\varepsilon^{b(m,\,n)}\,\{x-(m,n)\}^{-1};$$

and for $b = 0$,

$$\varepsilon^{\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma(-1)^{-m-n}\,q^{n^2}\,\{x-(m,n)\}^{-1} \;\ldots\ldots (45),$$
$$\varepsilon^{-\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma(-1)^{-m-n}\,q_1^{m^2}\,\{x-(m,n)\}^{-1}.$$

Next the system,

$$\mathfrak{G}x \div \gamma x = \Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,n)\}^{-1} \;\ldots\ldots\; (B').$$
$$gx \div \gamma x = \Sigma\Sigma\,(-1)^{m}\,\{x-(m,n)^{-1},$$
$$\mathfrak{G}x \div \gamma x = \Sigma\Sigma\,(-1)^{n}\,\{x-(m,n)\}^{-1};$$

$$\gamma x \div gx = -\,b^{-1}\,c^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(\overline{m},n)\}^{-1},$$
$$Gx \div gx = -\,c^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\overline{m},n)\}^{-1},$$
$$\mathfrak{G}x \div gx = -\,b^{-1}\,\Sigma\Sigma\,(-1)^{m}\,\{x-(\overline{m},n)\}^{-1};$$

$$\gamma x \div Gx = -\,b^{-1}\,e^{-1}\,\Sigma\Sigma\,(-1)^{m}\,\{x-(m,\overline{n})\}^{-1},$$
$$gx \div Gx = i\,e^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,\overline{n})\}^{-1},$$
$$\mathfrak{G}x \div Gx = i\,b^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(m,\overline{n})\}^{-1};$$

$$\gamma x \div \mathfrak{G}x = i\,c^{-1}\,e^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\overline{m},\overline{n})\}^{-1},$$
$$gx \div \mathfrak{G}x = e^{-1}\,\Sigma\Sigma\,(-1)^{m}\,\{x-(\overline{m},\overline{n})\}^{-1},$$
$$Gx \div \mathfrak{G}x = i\,c^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(\overline{m},\overline{n})\}^{-1}.$$

which is partially given by Abel.

We may obtain, in like manner, expressions for the functions

$$\frac{1}{\gamma x\, gx},\ \frac{1}{\gamma x\, Gx},\ \ldots \text{ (six terms of this form) } \ldots\ (C'),$$

$$\frac{Gx}{\gamma x\, gx},\ \ldots \qquad \text{(twelve)} \qquad \ldots\ (D'),$$

$$\frac{\gamma x\, gx}{\mathfrak{G}x\, Gx},\ \ldots \qquad \text{(six)} \qquad \ldots\ (E'),$$

$$\frac{1}{\gamma x\, gx\, Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\ (F'),$$

$$\frac{\mathfrak{G}x}{\gamma x\, gx\, Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\ (G'),$$

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x},\ \ldots \qquad \text{(one)} \qquad \ldots\ (H');$$

each of them, except (E'), (the system for which, admitting no exponential, has already been given,) multiplied by an exponential $\varepsilon^{\frac{1}{2}ax^2+bx}$, the limits of (a) being $\pm 2\beta$, $\pm\beta$, $-$, $\pm 3\beta$, $\pm 2\beta$, $\pm 4\beta$. For the limiting values, b must be entirely impossible for the superior limit, and entirely possible for the inferior one.

Thus the last case is

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x}\,\varepsilon^{\frac{1}{2}ax^2+bx} \;\ldots\ldots\ldots\ldots (H'),$$

$$= \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}a(m,\,n)^2+b\,(m,\,n)}\, q_1^{\;2m^2}\, q^{2n^2}\left\{x-(m,\,n)\right\}^{-1}\right]$$

$$- \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}a(\bar{m},\,n)^2+b\,(\bar{m},\,n)}\, q_1^{\;2(m+\frac{1}{2})^2}\, q^{2n^2}\left\{x-(\bar{m},\,n)\right\}^{-1}\right]$$

$$+ \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}a(m,\,\bar{n})^2+b\,(m,\,\bar{n})}\, q_1^{\;2m^2}\, q^{2(n+\frac{1}{2})^2}\left\{x-(m,\,\bar{n})\right\}^{-1}\right]$$

$$- \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}a(\bar{m},\,\bar{n})^2+b\,(\bar{m},\,\bar{n})}\, q_1^{\;2(m+\frac{1}{2})^2}\, q^{2(n+\frac{1}{2})^2}\left\{x-(\bar{m},\,\bar{n})\right\}^{-1}\right];$$

in particular

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x}\,\varepsilon^{2\beta x^2} = \Sigma\Sigma\, q_1^{\;4m^2}\left\{x-(m,\,n)\right\}^{-1} \;\ldots\ldots (46),$$

$$- \Sigma\Sigma\, q_1^{\;(2m+1)^2}\left\{x-(\bar{m},\,n)\right\}^{-1}$$

$$+ \Sigma\Sigma\, q_1^{\;4m^2}\left\{x-(\bar{m},\,n)\right\}^{-1}$$

$$+ \Sigma\Sigma\, q_1^{\;(2m+1)^2}\left\{x-(\bar{m},\,\bar{n})\right\}^{-1},$$

or the analogous formula obtained by changing β, q_1, m into $-\beta$, q, n.

The function $\phi^2 x$, which is even, and for which $\lambda = 0$, cannot be expanded entirely in a series of partial fractions; but $(x-a)^{-1}\,\phi^2 x$ may be so expanded. Multiply by $(x-a)$, the second side has for its general term

$$(x-a)\,(Mx+N)\,\{x-(\bar{m},\,\bar{n})^{-2},$$

equivalent to

$$K' + (M'x+N')\,\{x-(\bar{m},\,\bar{n})\}^{-2}.$$

Summing all the K''s, we have an equation of the form

$$\phi^2 x = A + \Sigma\Sigma\left[L\left\{x-(\bar{m},\,\bar{n})\right\}^{-2} + M\left\{x-(\bar{m},\,\bar{n})\right\}^{-1}\right] \ldots (47).$$

To determine the coefficients as simply as possible, change x into $x+\frac{1}{2}\omega+\frac{1}{2}n\omega i$,

$$-\,e^{-2}\,c^{-2}\,\overline{\phi x}|^{-2} = A + \Sigma\Sigma\left[L\left\{x-(m,\,n)\right\}^{-2} + M\left\{x-(m,\,n)\right\}^{-1}\right] \ldots, (48),$$

$$L = -\,e^{-2}\,c^{-2}\left[\{x-(m,\,n)\}^2\,\overline{\phi x}|^{-2}\right] \ldots\ldots\ldots\ldots (49),$$

$$M = -\,e^{-2}\,c^{-2}\,d_x\left[\{x-(m,\,n)\}^2\,\overline{\phi x}|^{-2}\right], \quad x=(m,\,n).$$

Or writing $x+(m,\,n)$ for x,

$$L = -\,e^{-2}\,c^{-2}\,(x^2\overline{\phi x}|^{-2}), \quad (x=0), \qquad L = e^{-2}\,c^{-2} \ldots (50),$$

$$M = -\,e^{-2}\,c^{-2}\,d_x\,(x^2\overline{\phi x}|^{-2}) = 0;$$

$$\phi^2 x = \;A - e^{-2}\,c^{-2}\,\Sigma\Sigma\left\{x-(\bar{m},\,\bar{n})\right\}^{-2} \ldots\ldots\ldots\ldots (51).$$

Integrating twice,

$$\int_0 dx \int_0 dx \,.\, \phi^2 x = \tfrac{1}{2} A x^2 + e^{-2} c^{-2}\, \Sigma\Sigma\, l\,\{x - (\overline{m}, \overline{n})\} \ldots . (52),$$

or

$$\mathfrak{G}x = \varepsilon^{-\frac{1}{2}e^2c^2Ax^2 + e^2c^2 \int_0 dx \int_0 dx \,.\, \phi^2 x} \ldots\ldots\ldots (53),$$

an equation from which it is easy to determine the coefficient A.

Suppose for a moment $\phi_{\prime} x = \int_0 \phi^2 x\, dx$, $\phi_{\prime\prime} x = \int_0 \phi_{\prime} x\, dx$; then, since $\phi^2 (x + \omega) - \phi^2 x = 0$,

$$\phi_{\prime} (x + \omega) = \phi x = \phi_{\prime} \omega, \quad \phi_{\prime\prime} (x + \omega) - \phi_{\prime\prime} x = \phi_{\prime\prime} \omega + x \phi_{\prime} \omega.$$

But similarly $\phi^2 x - \phi^2 (\omega - x) = 0$; whence

$$\phi_{\prime} x + \phi_{\prime} (\omega - x) = \phi_{\prime} \omega, \quad \phi_{\prime\prime} x - \phi_{\prime\prime} (\omega - x) + \phi_{\prime\prime} \omega = x \phi_{\prime} \omega\,;$$

whence, writing $x = \dfrac{\omega}{2}$,

$$\phi_{\prime} \omega = 2 \phi_{\prime} \frac{\omega}{2}, \quad \phi_{\prime\prime} \omega = \omega . \phi_{\prime} \frac{\omega}{2}, \text{ or } \phi_{\prime\prime} (x + \omega) - \phi_{\prime\prime} x = \phi_{\prime} \left(\frac{\omega}{2}\right)(2x + \omega).$$

Hence $\mathfrak{G}(x + \omega) = \varepsilon^{-\frac{1}{2}e^2c^2\left(A - \frac{\omega}{2}\phi_{\prime}\frac{\omega}{2}\right)(2\omega x + \omega^2)}\, \mathfrak{G}x \ldots . (54).$

But $\mathfrak{G}(x + \omega) = \varepsilon^{\beta\omega x}\, q_1^{-\frac{1}{2}}\, \mathfrak{G}x = \varepsilon^{\frac{\beta}{2}(2\omega x + \omega^2)}\, \mathfrak{G}x \ldots . (55);$

or, comparing these,

$$- e^2 c^2 \left(A - \frac{2}{\omega} \phi_{\prime} \frac{\omega}{2}\right) = \beta \ldots\ldots\ldots (56),$$

$$- \tfrac{1}{2} e^2 c^2 A = \tfrac{1}{2}\beta - \frac{e^2 c^2}{\omega} \phi_{\prime} \frac{\omega}{2} \ldots\ldots\ldots (57),$$

or writing

$$M = \frac{e^2 c^2}{\omega} \int_0^{\frac{\omega}{2}} \phi^2 x\, dx \ldots\ldots\ldots\ldots (58),$$

$$\mathfrak{G}x = \varepsilon^{(\frac{1}{2}\beta - M)\, x^2 + e^2c^2 \int_0 dx \int_0 dx \,.\, \phi^2 x} \ldots\ldots (I');$$

which is the formulæ corresponding to the one of Jacobi's referred to at the beginning of this paper. Analogous formulæ may be deduced from it by writing $x + \dfrac{\omega}{2}$, or $x + \dfrac{\upsilon i}{2}$, or $x + \dfrac{\omega}{2} + \dfrac{\upsilon i}{2}$, instead of x.

The following formulæ, making the necessary changes of notation, are taken from Jacobi. We have

$$\phi^2 (x + a) - \phi^2 (x - a) = \frac{4 \phi a . f a . F a . \phi x . f x F x}{(1 + e^2 c^2 . \phi^2 a . \phi^2 x)^2} \ldots\ldots (59),$$

whence

$$\int_0 \{\phi^2 (x + a) - \phi^2 . (x - a)\}\, dx = \frac{2 \phi a . f a . F a . \phi^2 x}{1 + e^2 c^2 \phi^2 . a . \phi^2 x} \ldots (60).$$

The first side of which is

$$\int_{-a}.\phi^2.(x+a)\,dx - \int_a.\phi^2(x-a)\,dx - 2.\int_0\phi^2 a.da \ldots (61).$$

Hence, multiplying by e^2c^2, and observing the value of $\mathfrak{G}x$,

$$\frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} - \frac{\mathfrak{G}'(x-a)}{\mathfrak{G}(x-a)} - 2\,\frac{\mathfrak{G}'a}{\mathfrak{G}a} = \frac{2e^2c^2.fa.Fa.\phi a.\phi^2 x}{1+e^2c^2\phi^2 a.\phi^2 x} \ldots (62).$$

If in this case we interchange x, a and add,

$$\frac{\mathfrak{G}'x}{\mathfrak{G}x} + \frac{\mathfrak{G}'a}{\mathfrak{G}a} - \frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} = e^2c^2.\phi a.\phi x.\phi(a+x) \ldots (63).$$

[By subtracting, we should have obtained an equation only differing from the above in the sign of (a)].

Integrating the last equation but one, with respect to (a),

$$l\mathfrak{G}(x+a) + l\mathfrak{G}(x-a) - 2l\mathfrak{G}x - 2l\mathfrak{G}a = l(1+e^2c^2\phi^2 x\phi^2 a).$$

the integral being taken from $a = 0$. Whence

$$\mathfrak{G}(x+a)\,\mathfrak{G}(x-a) = \mathfrak{G}^2 x\mathfrak{G}^2 a\,(1+e^2c^2\phi^2 x\phi^2 a) \ldots (64).$$

Or $\quad \mathfrak{G}(x+a)\,\mathfrak{G}(x-a) = \mathfrak{G}^2 x.\mathfrak{G}^2 a + e^2c^2.\gamma^2 x\gamma^2 a,$

whence also

$$\left.\begin{aligned} \gamma(x+a)\,\gamma(x-a) &= \gamma^2 x\mathfrak{G}^2 a - \gamma^2.a\mathfrak{G}^2 x. \\ g(x+a)\,g(x-a) &= g^2 x\mathfrak{G}^2 x - c^2.g^2 a.\mathfrak{G}^2 x. \end{aligned}\right\} (J').$$

$$G(x+a).\,G(x-a) = G^2 x.\mathfrak{G}^2 a + e^2 G^2 a.\mathfrak{G}^2 x.$$

These equations being obtained from the first by the change of x into $x + \frac{\omega}{2}$, $x + \frac{\upsilon i}{2}$, $x + \frac{\omega}{2} + \frac{\upsilon i}{2}$. They form a most important group of formulæ in the present theory. (By integrating the same formulæ with respect to x, and representing by $\Pi(x, a)$ the integral $\int_0 \frac{-e^2c^2\phi a fa Fa\phi^2 x dx}{1+e^2c^2\phi^2 a\phi^2 x}$, Jacobi obtains

$$\Pi(x, a) = \tfrac{1}{2}l\,\frac{\mathfrak{G}(x-a)}{\mathfrak{G}(x+a)} + x\,\frac{\mathfrak{G}'a}{\mathfrak{G}a}:$$

an equation which conducts him almost immediately to the formulæ for the addition of the argument or of the parameter in the function Π. This, however, is not very closely connected with the present subject. For some formulæ also deduced from (63), by which $\frac{\mathfrak{G}(x-a)\,\mathfrak{G}(y-a)\,\mathfrak{G}(x+y+a)}{\mathfrak{G}(x+a)\,\mathfrak{G}(y+a)\,\mathfrak{G}(x+y-a)}$ is expressed in terms of the function ϕ. See Jacobi.

NOTE.—We have

$$\gamma_\beta x = x\Pi\Pi\left(1 + \frac{x}{(m, n)}\right).$$

$$g\beta x = \Pi\Pi\left(1 + \frac{x}{(\overline{m}, n)}\right).$$

the limits of n being $\pm q$, and those of m being $\pm p$, in the first case, and $p, -p-1$, in the second case. Also $\frac{p}{q} = \infty$. We deduce immediately

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = \left(x + \frac{\omega}{2}\right)\Pi\Pi\left\{1 + \frac{\left(x + \frac{\omega}{2}\right)}{(m, n)}\right\}$$

$$= \Pi\Pi.\left(1 + \frac{x}{(\overline{m}, n)}\right) \div \frac{\omega}{2}\,\Pi\Pi\,\frac{(m, n)}{(\overline{m}, n)};$$

(paying attention to the omission of $(m = 0, n = 0)$ in $\gamma_\beta x$, and supposing that this value enters into the numerator of the expression just obtained, but not into its denominator). This is of the form

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = A\Pi\Pi\left(1 + \frac{x}{(\overline{m}, n)}\right);$$

but the limits are not the same in this product and in $g_\beta x$. In the latter m assumes the value $-p-1$, which it does not in the former. Hence

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) \div g_\beta x = A \div \Pi_n\left(1 + \frac{x}{-(p + \frac{1}{2})\,\omega + nvi}\right).$$

And the above product reduces itself to unity in consequence of all the values assumed by n being indefinitely small compared with the quantity $(p + \frac{1}{2})$; we have therefore

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = Ag_\beta x \ldots\ldots\ldots\ldots (27),$$

and similar expressions for the remaining functions. To illustrate this further, suppose we had been considering, instead of $\gamma_\beta x$ the function $\gamma_{-\beta} x$, given by the same formulæ, only instead of $\frac{p}{q} = \infty$, $\frac{p}{q} = 0$. We have in this case also

$$\gamma_{-\beta}\left(x + \frac{\omega}{2}\right) \div g_{-\beta} x = A' \div \Pi_n\left(1 + \frac{x}{(-p + \frac{1}{2})\,\omega + nvi}\right).$$

A' different from A on account of the different limits. The divisor of the second side takes the form

$$\{x-(p+\tfrac{1}{2})\omega.\}.\Pi\left(1+\frac{x-(p+\frac{1}{2})\omega}{nvi}\right)\div(-p+\tfrac{1}{2})\omega.\Pi\left(1-\frac{(p+\frac{1}{2})\omega}{nvi}\right),$$

and the extreme values of n being infinite as compared with p. This may be reduced to

$$-\sin\frac{\pi}{vi}\{x-(p+\tfrac{1}{2})\omega\}\div\sin(p+\tfrac{1}{2})\omega.$$

$$=\varepsilon^{\frac{\pi}{v}[(p+\frac{1}{2})\omega-x]}\div\varepsilon^{\frac{\pi}{v}[(p+\frac{1}{2})\omega]}=\varepsilon^{-\frac{\pi x}{v}};$$

neglecting the exponentials whose indices are infinitely great and negative. Observing the value of β this becomes $\varepsilon^{-\omega\beta x}$, and we have

$$\gamma_{-\beta}\left(x+\frac{\omega}{2}\right)=\varepsilon^{\beta\omega x}.A'g_{-\beta}.x:$$

a result of the form of that which would be deduced from the equations $\gamma_{-\beta}x=\varepsilon^{\beta x^2}\gamma_\beta x$, $g_{-\beta}x=\varepsilon^{\beta x^2}g_\beta x$, $\gamma_\beta\left(x+\frac{\omega}{2}\right)=Ag\beta x.$

It is scarcely necessary to remark that $\gamma_{-\beta}x$ has the same relations to the change of x into $x+\frac{vi}{2}$ as $\gamma_\beta x$ has to that of x into $x+\frac{\omega}{2}$.

VI.—ON BRIANCHON'S HEXAGON.

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THE following is a proof of Brianchon's property of the hexagon circumscribing a conic section, in which the method of multiplication is used, which I made use of to solve a problem in transversals in No. XXI. p. 113. The property is stated in the last number, and proved for each case separately.

Let the conic sections be referred to two opposite sides of the circumscribing hexagon as axes, and let the equation to the curve be
$$ax+by-1=2B\sqrt{(xy)}.$$

Then, if $\alpha x+\beta y=1$ be the equation to a tangent,
$$ax+by-(\alpha x+\beta y)=2B\sqrt{(xy)}$$

must give equal values to $\sqrt{\frac{y}{x}}$;

hence
$$(a-\alpha)(b-\beta)=B^2 \quad \ldots\ldots\ldots (1).$$

Let AB, CD (see fig. 2) be the two sides taken as axes, AE, EC and BF, FD pairs of contiguous sides.

Let the equations be

$$\left.\begin{aligned} ax + \beta' y &= 1 \quad \text{to } AE \\ a'x + \beta y &= 1 \quad \text{"} \ \ EC \end{aligned}\right\} \ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} \gamma x + \delta' y &= 1 \quad \text{"} \ \ BF \\ \gamma' x + \delta y &= 1 \quad \text{"} \ \ FD \end{aligned}\right\} \ldots\ldots\ldots (3),$$

multiply equations (2) by h and k respectively, and add; therefore at E we have the relation

$$(ha + ka')\, x + (h\beta' + k\beta)\, y = h + k.$$

Similarly at F we have the relation

$$(h'\gamma + k'\gamma')\, x + (h'\delta' + k'\delta)\, y = h' + k' \ \ldots . (4);$$

and if the arbitrary multipliers be so chosen as to make these coincide, either will be the equation to the said line EF. In this case

$$ha + ka' = h'\gamma + k'\gamma',$$
$$h\beta' + k\beta = h'\delta' + k'\delta,$$
$$h + k = h' + k';$$

therefore, eliminating by cross multiplication h and k,

$$\left.\begin{aligned} 0 &= k' \left\{\gamma'(\beta - \beta') + \delta(a - a') + a'\beta' - a\beta\right\} \\ &\quad + h' \left\{\gamma(\beta - \beta') + \delta'(a - a') + a'\beta' - a\beta\right\} \end{aligned}\right\} \ldots (5).$$

Now by (1) we have the relations

$$(a - a).(b - \beta') = (a - a').(b - \beta) = (a - \gamma').(b - \delta) = \ldots$$

whence
$$\frac{\beta - \beta'}{a - a'} = \frac{b - \beta}{a - a} \ldots\ldots\ldots\ldots (6),$$

and
$$\frac{\delta - \beta'}{\gamma' - a} = \frac{\delta - b}{a - a};$$

therefore adding these equations we obtain

$$\frac{\gamma'(\beta - \beta') + \delta(a - a') + a'\beta' - a\beta}{(a - a')(\gamma' - a)} = \frac{\delta - \beta}{a - a}.$$

Similarly, by the symmetry of (1), changing γ and a and a into δ, β and b respectively, and *vice versâ*,

$$\frac{\delta'(a - a') + \gamma(\beta - \beta') + a'\beta' - a\beta}{(\beta - \beta')(\delta' - \beta)} = \frac{\gamma - a}{b - \beta};$$

therefore we have by (5) and (6)

$$0 = k'(\delta - \beta)(\gamma' - a) + h'(\delta' - \beta)(\gamma - a) \ldots . (7).$$

Now $\quad \alpha x + \delta y = 1$ is the equation to AD,

$\gamma x + \beta y = 1$ " " " BC;

therefore, multiplying by m and n, and adding,

$$(m\alpha + n\gamma)\, x + (m\delta + n\beta)\, y = m + n \;\ldots\ldots\; (8)$$

is a relation which holds at G, the intersection.

Assume m and n so as to satisfy the equations

$$m\alpha + n\gamma = h'\gamma + k'\gamma',$$
$$m\delta + n\beta = h'\delta' + k'\delta;$$

therefore $\qquad m(\gamma\delta - \alpha\beta) = h'(\gamma\delta' - \beta\gamma) + k'(\gamma\delta - \beta\gamma');$

or $\qquad (m - k')(\gamma\delta - \alpha\beta) = h'\gamma(\delta' - \beta) + k'\beta(\alpha - \gamma').$

Similarly $(n - h')(\gamma\delta - \alpha\beta) = h'\alpha(\beta - \delta') + k'\delta(\gamma' - \alpha);$

hence $\{m + n - (h' + k')\}(\gamma\delta - \alpha\beta)$

$$= h'(\gamma - \alpha)(\delta' - \beta) + k'(\delta - \beta)(\gamma' - \alpha) = 0, \text{ by } (7);$$

and $$m + n = h' + k';$$

therefore the relations (8) and (4) coincide, or the point G is in the straight line EF; which proves the proposition.

VII.—ON THE LINES OF CURVATURE OF SURFACES OF THE SECOND ORDER.

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THE method usually followed in works on Geometry of Three Dimensions, in treating the differential equation to the lines of curvature of an ellipsoid, leads to an unsymmetrical integral, involving only two of the co-ordinates, and therefore representing the projection of the lines of curvature on one of the co-ordinate planes. In the following paper, by making use of an equivalent process, but preserving the symmetry with respect to the two variables which are involved, an integral is obtained which enables us from the symmetry to infer the equations to the projections on the other two co-ordinate planes. By combining the three forms of the integral thus obtained, we arrive at the integral given by Mr. Ellis, and at other symmetrical formulæ.

Let the equation to the surface be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The differential equation to the lines of curvature is consequently

$$\frac{(b^2 - c^2)\, x}{dx} + \frac{(c^2 - a^2)\, y}{dy} + \frac{(a^2 - b^2)\, z}{dz} = 0.$$

Let $\frac{x^2}{a^2} = u$, $\frac{y^2}{b^2} = v$, $\frac{z^2}{c^2} = w$. The preceding equations become

$$u + v + w = 1 \ldots\ldots\ldots\ldots (1),$$

$$\frac{(b^2 - c^2)\,u}{du} + \frac{(c^2 - a^2)\,v}{dv} + \frac{(a^2 - b^2)\,w}{dw} = 0 \ldots. (2).$$

Eliminating u from the latter equation, by means of the former, we have

$$\frac{b^2 - c^2}{du} + v\left(\frac{c^2 - a^2}{dv} - \frac{b^2 - c^2}{du}\right) + w\left(\frac{a^2 - b^2}{dw} - \frac{b^2 - c^2}{du}\right) = 0,$$

or
$$b^2 - c^2 = \frac{v\,(-c^2 du - c^2 dv + a^2 du + b^2 dv)}{dv}$$
$$- \frac{w\,(-b^2 du - b^2 dw + a^2 du + c^2 dw)}{dw}.$$

But $du + dv + dw = 0$, by (1); and hence the equation to the lines of curvature may be put under the form

$$\frac{v}{dv} - \frac{w}{dw} = \frac{b^2 - c^2}{a^2 du + b^2 dv + c^2 dw}.$$

If from this equation we eliminate du, we obtain an equation of Clairaut's form, of which the integral is found by substituting for $\frac{dv}{dw}$ an arbitrary constant. For the sake of symmetry we shall denote this constant by $\frac{g}{h}$; and we must consequently substitute g and h for dv and dw in the differential equation, and therefore also for du, $-(g + h)$, which we shall denote by f. Thus we have

$$\left.\begin{aligned} &\frac{du}{f} = \frac{dv}{g} = \frac{dw}{h} \\ \text{where}\quad &f + g + h = 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots (3),$$

and the complete integral is

$$\frac{v}{g} - \frac{w}{h} = \frac{b^2 - c^2}{a^2 f + b^2 g + c^2 h} \ldots\ldots\ldots (4).$$

Also, by the symmetry, we have for the integrals involving the variables wu and uv,

$$\left.\begin{aligned} \frac{w}{h} - \frac{u}{f} &= \frac{c^2 - a^2}{a^2 f + b^2 g + c^2 h} \\ \frac{u}{f} - \frac{v}{g} &= \frac{a^2 - b^2}{a^2 f + b^2 g + c^2 h} \end{aligned}\right\} \ldots\ldots\ldots (4),$$

The manner in which the quantities f, g, h have been introduced, shows clearly how they represent only one arbitrary constant. Any one of the equations (4) may be written in such a form as to contain only one arbitrary constant explicitly; and it will be shown below how f, g, h may be expressed symmetrically by two arbitrary constants, one of which is irrelevant, as it enters as a factor in the integral.

From the equations (4), as from the ordinary forms, the properties of the projections of the lines of curvature may be readily deduced. Thus, taking the second, and substituting for w, u, and g their values $\frac{z^2}{c^2}$, $\frac{x^2}{a^2}$, and $-(f+h)$, we have

$$\frac{z^2}{c^2 h} - \frac{x^2}{a^2 f} = \frac{a^2 - c^2}{(b^2 - c^2)h - (a^2 - b^2)f}.$$

Let a^2, b^2, c^2 be positive quantities in descending order of magnitude. Then, unless f and h have opposite signs, this equation cannot be satisfied by any values of z, x which satisfy the inequality

$$\frac{z^2}{c^2} + \frac{x^2}{a^2} < 1;$$

that is, by values which correspond to any point of the ellipsoid. Hence we may write the equation as

$$\frac{z^2}{\gamma^2} + \frac{x^2}{\alpha^2} = 1 \ldots\ldots\ldots\ldots\ldots (5),$$

where
$$\gamma^2 = \frac{c^2(a^2 - c^2)h}{(b^2 - c^2)h - (a^2 - b^2)f},$$

$$\alpha^2 = -\frac{a^2(a^2 - c^2)f}{(b^2 - c^2)h - (a^2 - b^2)f}.$$

Eliminating $f : h$ between these equations, we have

$$\frac{\gamma^2(b^2 - c^2)}{c^2} + \frac{\alpha^2(a^2 - b^2)}{a^2} = a^2 - c^2 \ldots\ldots (6).$$

Hence we conclude that the projections of the lines of curvature on the plane of the greatest and least axes of the ellipsoid, are the ellipses whose semiaxes, γ, α, are connected by the equation (6). Thus the construction for describing them is as follows. Draw an ellipse, concentric with the ellipsoid, in the plane ca, with the lines

$$c\left(\frac{a^2 - c^2}{b^2 - c^2}\right)^{\frac{1}{2}}, \quad a\left(\frac{a^2 - c^2}{a^2 - b^2}\right)^{\frac{1}{2}},$$

as semiaxes. Take any point in this ellipse, draw perpendiculars to the axes, and with the intersections as vertices,

describe a concentric ellipse. This will be the projection of a line of curvature. Also, by giving the point assumed in the auxiliary ellipse every possible position in its circumference, we obtain the projections of all the lines of curvature. Similar constructions are applicable to the projections of the lines of curvature on the other two principal planes; and, by taking one or two of the quantities a^2, b^2, c^2 negative, we may extend the rules to hyperboloids of one or of two sheets.

In the case we have taken, of an ellipsoid, and the plane of the greatest and least axes for the plane of projection, each curve intersects the consecutive one, and the locus of these intersections may be found from (5) and (6) by the ordinary process. Thus, by differentiation,

$$\frac{z^2}{\gamma^3}\,d\gamma + \frac{x^2}{\alpha^3}\,d\alpha = 0,$$

$$\frac{a^2-b^2}{a^2}\,\alpha\,d\alpha + \frac{b^2-c^2}{c^2}\,\gamma\,d\gamma = 0.$$

Hence $$\frac{c^2}{b^2-c^2}\,\frac{z^2}{\gamma^4} = \frac{a^2}{a^2-b^2}\,\frac{x^2}{\alpha^4},$$

which gives $$\frac{\gamma^2}{cz\,(b^2-c^2)^{-\frac{1}{2}}} = \frac{\alpha^2}{ax\,(a^2-b^2)^{-\frac{1}{2}}}.$$

By combining this equation, first with (6) and then with (5), we find each member

$$= \frac{a^2-c^2}{\frac{z}{c}(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2-b^2)^{\frac{1}{2}}} = \frac{z}{c}(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2-b^2)^{\frac{1}{2}}.$$

Hence $$\frac{z}{c}(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2-b^2)^{\frac{1}{2}} = (a^2-c^2)^{\frac{1}{2}} \; \ldots\ldots (7),$$

is the equation to the required locus, which is therefore a group of four straight lines (on account of the double signs of the radicals) forming a rhombus, of which the diagonals coincide with the axes of c and a.

Thus we see that the projections of the lines of curvature on the plane of the greatest and least axes, are ellipses inscribed in a rhombus, with their axes coincident with those of the ellipsoid. If we consider b^2 as not of intermediate magnitude between a^2 and c^2, the equation (7) represents an imaginary group of straight lines, which shows that the projection of any line of curvature on the plane of the greatest and mean, or of the mean and least axes, does not meet its consecutive.

It may be remarked with respect to equations (4), that any one of them may be deduced from any other, by combining it with the equation $u + v + w = 1$, as is easily verified. Also, by multiplying the first by a^2, the second by b^2, and the third by c^2, and adding, we have

$$\frac{u\,(b^2 - c^2)}{f} + \frac{v\,(c^2 - a^2)}{g} + \frac{w\,(a^2 - b^2)}{h} = 0 \;\ldots\ldots\; (8),$$

which is the symmetrical integral given by Mr. Ellis, (Vol. II. p. 133). This equation might also have been found directly, when it was proved that

$$\frac{du}{f} = \frac{dv}{g} = \frac{dw}{h};$$

by eliminating by means of these relations, du, dv, dw from (2), the differential equation to the lines of curvature, which is the method followed by Mr. Ellis.

Without losing generality, we may substitute for f, g, h, any expressions in terms of two distinct constants which satisfy the condition $f + g + h = 0$. Thus, if we take k and υ for the constants, we may assume

$$f = ka^2\,(b^2 - c^2) - k\upsilon\,(b^2 - c^2),$$

or

$$\left.\begin{aligned} f &= k\,(b^2 - c^2)\,(a^2 - \upsilon), \\ g &= k\,(c^2 - a^2)\,(b^2 - \upsilon), \\ h &= k\,(a^2 - b^2)\,(c^2 - \upsilon), \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (9).$$

Making these substitutions in (8), we have

$$\left.\begin{aligned} &\frac{u}{a^2 - \upsilon} + \frac{v}{b^2 - \upsilon} + \frac{w}{c^2 - \upsilon} = 0, \\ \text{or}\quad &\frac{x^2}{a^2\,(a^2 - \upsilon)} + \frac{y^2}{b^2\,(b^2 - \upsilon)} + \frac{z^2}{c^2\,(c^2 - \upsilon)} = 0, \end{aligned}\right\} \;\ldots\ldots\; (10).$$

Adding this equation, multiplied by υ, to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \;\ldots\ldots\ldots\ldots\; (a),$$

we have

$$\frac{x^2}{a^2 - \upsilon} + \frac{y^2}{b^2 - \upsilon} + \frac{z^2}{c^2 - \upsilon} = 1 \;\ldots\ldots\ldots\; (11).$$

This equation shows that the lines of curvature of any surface of the second order with a centre, are its intersections with confocal surfaces of the second order. Since this property is independent of the centre, it follows that it must also be true for the case of a surface without a centre.

If the coordinates x, y, z, of any point in the line of curvature be given, by substituting these values for x, y, z

in (10), which will be a quadratic in v, we may determine two values of this parameter, which, substituted in (11), will give the equations to the hyperboloids of one sheet and of two sheets confocal with the ellipsoid, which cut it in the two lines of curvature passing through the given point x_1, y_1, z_1. In general, equation (11) may be considered as a cubic for determining v, when x, y, z have any given real values whatever, x_1, y_1, z_1. The three roots, which may readily be shown to be real, correspond to the three species of surfaces confocal with the ellipsoid (a, b, c) which intersect in the point x_1, y_1, z_1. In the present case, when this point is on the surface of the ellipsoid (a, b, c), and therefore x_1, y_1, z_1 satisfy the equation (a), one root of the cubic is zero, and the other two are the roots of the quadratic (10), which is the *reduced equation*. Thus, whether we take the form (10) or (11) of the integral, the two arbitrary constants may be determined by the solution of a quadratic equation, from the condition that the curve passes through a given point.

If we wish to determine the direction of the tangent at any point of a line of curvature, we may follow the usual process of differentiation for curves of double curvature.

Thus, taking any two of the equations (4), we find

$$\frac{du}{f} = \frac{dv}{g} = \frac{dw}{h}.$$

But if l, m, n be the direction-cosines of the tangent, we have

$$\frac{l}{\frac{a^2}{x}du} = \frac{m}{\frac{b^2}{y}dv} = \frac{n}{\frac{c^2}{z}dw};$$

and hence

$$\frac{l}{\frac{a^2}{x}f} = \frac{m}{\frac{b^2}{y}g} = \frac{n}{\frac{c^2}{z}h} \quad \ldots\ldots\ldots (12).$$

If we substitute for f, g, h their values by (9), these equations become

$$\frac{l}{\frac{a^2}{x}(b^2-c^2)(a^2-v)} = \frac{m}{\frac{b^2}{y}(c^2-a^2)(b^2-v)} = \frac{n}{\frac{c^2}{z}(a^2-b^2)(c^2-v)} \ldots (13).$$

We may also determine l, m, n by the ordinary formulæ for the principal directions of curvature. Thus, if ρ be the radius of curvature of the normal section touching the line of curvature at the point considered, we have

$$l = \frac{\mu x}{a^2 - p\rho}, \quad m = \frac{\mu y}{b^2 - p\rho}, \quad n = \frac{\mu z}{c^2 - p\rho} \quad \ldots . (14);$$

and $p\rho$ is one of the values of Q deduced from the quadratic equation

$$\frac{x^2}{a^2(a^2-Q)}+\frac{y^2}{b^2(b^2-Q)}+\frac{z^2}{c^2(c^2-Q)}=0 \;\ldots\ldots\; (15).$$

Also υ is a root of this equation, since, when it is substituted for Q, the equation is identical with (10), one of the equations of the line of curvature considered. Thus a line of curvature is the locus of points on the surface, for which one root of (15) is constant. Now, by combining equations (13) and (14), we have

$$\left.\begin{aligned} &\frac{a^2(b^2-c^2)(a^2-\upsilon)(a^2-p\rho)}{x^2} \\ =&\frac{b^2(c^2-a^2)(b^2-\upsilon)(b^2-p\rho)}{y^2} \\ =&\frac{c^2(a^2-b^2)(c^2-\upsilon)(c^2-p\rho)}{z^2} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (16).$$

These equations show that υ and $p\rho$ cannot both be constant, and therefore they must be different roots of the equation (15). Hence, if ρ' be the radius of curvature of a normal section perpendicular to the line of curvature at P, we have

$$\upsilon = p\rho';$$

which shows that the radii of curvature of sections perpendicular to a line of curvature at different points, are inversely proportional to the perpendiculars from the centre upon the tangent planes at those points.

The equations (16) may be verified directly, since, $p\rho$ and υ being the two roots of (15), we have

$$\upsilon \,.\, p\rho = a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right),$$

$$\upsilon + p\rho = \frac{(b^2+c^2)\,x^2}{a^2}+\frac{(c^2+a^2)\,y^2}{b^2}+\frac{(a^2+b^2)\,z^2}{c^2}.$$

Hence $(a^2-\upsilon)(a^2-p\rho)$

$$=a^4-\{a^2(b^2+c^2)-b^2c^2\}\frac{x^2}{a^2}-\{a^2(c^2+a^2)-c^2a^2\}\frac{y^2}{b^2}-\{a^2(a^2+b^2)-a^2b^2\}\frac{z^2}{c^2}$$

$$=a^4\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)-\{a^2(b^2+c^2)-b^2c^2\}\frac{x^2}{a^2}$$

$$=(a^2-b^2)(a^2-c^2)\frac{x^2}{a^2}.$$

Hence $\dfrac{a^2(b^2-c^2)(a^2-\upsilon)(a^2-p\rho)}{x^2}=-(b^2-c^2)(c^2-a^2))a^2-b^2);$

and therefore we infer that each member of (16) is equal to this expression, on account of the symmetry.

Let l', m', n' be the direction-cosines of the principal section corresponding to the root v of the quadratic equation (15). We shall have, by the formulæ which correspond to (14),

$$l' = \frac{\mu x}{a^2 - v}, \quad m' = \frac{\mu y}{b^2 - v}, \quad n' = \frac{\mu z}{c^2 - v} \quad \ldots\ldots (17).$$

Thus, by means of the same root of the quadratic equation, we have, in (13) and (17), expressed the direction-cosines of each of the two principal sections. If we put λ for each member of (13), these equations give

$$\begin{aligned} ll' + mm' + nn' &= \lambda\mu \left\{a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2)\right\} \\ &= 0, \end{aligned}$$

which proves that the principal directions of curvature are at right angles to one another; a theorem, of which many other different proofs have been given.

Dec. 1844.

VIII.—MATHEMATICAL NOTE.

1.—Light diverging from a point is incident on a given surface at a given point; find the direction of the reflected ray.

Let l, m, n be the direction cosines of the normal (N) at the point x, y, z,

$$\left.\begin{array}{l} a, \beta, \gamma \\ a', \beta', \gamma' \end{array}\right\} \text{be those of the} \left\{\begin{array}{l} \text{incident ray } (I), \\ \text{reflected ray } (R), \end{array}\right.$$

ι the angle of incidence;

then, because (N), (I), (R) are in the same plane (f, g, h), we have

$$\left.\begin{array}{r} fl + gm + hn = 0, \\ fa + g\beta + h\gamma = 0, \\ fa' + g\beta' + h\gamma' = 0, \end{array}\right\} \ldots\ldots\ldots (1);$$

hence, λ, μ being indeterminate multipliers,

$$l = \lambda a + \mu a', \quad m = \lambda\beta + m\beta', \quad n = \lambda\gamma + \mu\gamma' \ldots (2).$$

Also

$$\left.\begin{array}{r} \cos\iota = la + m\beta + n\gamma, \\ \cos\iota' = la' + m\beta' + n\gamma', \\ \cos 2\iota = aa' + \beta\beta' + \gamma\gamma', \end{array}\right\} \ldots\ldots\ldots (3);$$

hence, multiplying equations in (2) by l, m, n respectively, and adding, also by a, β, γ and a', β', γ', and subtracting the results, we get by (3)

$$1 = (\lambda + \mu)\cos\iota,$$

$$0 = \lambda - \mu,$$

or

$$\lambda = \mu = \frac{1}{2\cos\iota},$$

whence equations (2) give

$$\alpha' = 2l \cos \iota - \alpha,\quad \beta' = 2m \cos \iota - \beta,\quad \gamma' = 2n \cos \iota - \gamma \ \ldots (4),$$

or equations to the reflected ray (2) are

$$\frac{\xi - x}{2l \cos \iota - \alpha} = \frac{\eta - y}{2m \cos \iota - \beta} = \frac{\zeta - z}{2n \cos \iota - \gamma} \ \ldots . (5).$$

Cor. 1. Let the incident ray pass through the origin; and let p be the length of the perpendicular from the origin on the tangent plane, r the distance of the point of incidence from the origin: then

$$\cos \iota = \frac{p}{r},\quad \frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \frac{1}{r};$$

hence equations (5) become

$$\frac{\xi - x}{2lp - x} = \frac{\eta - y}{2mp - y} = \frac{\zeta - z}{2np - z} \ \ldots\ldots\ldots (6).$$

Ex. Let the surface be an ellipsoid, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

then $$l = \frac{px}{a^2},\quad m = a,\quad n = a,$$

and $$\frac{\xi - x}{\left(2\frac{p^2}{a^2} - 1\right)x} = \frac{\eta - y}{\left(2\frac{p^2}{b^2} - 1\right)y} = \frac{\zeta - z}{\left(2\frac{p^2}{c^2} - 1\right)z} \ \ldots (7).$$

Cor. 2. When the reflected ray passes through the line,

$$\frac{\xi}{l_1} = \frac{\eta}{m_1} = \frac{\zeta}{n_1} = \rho \text{ suppose } \ldots\ldots\ldots (7),$$

we have, substituting in (6), and calling each fraction q,

$$\left.\begin{aligned} l_1\rho - (2lp - x)\, q &= x, \\ m_1\rho - (2mp - y)\, q &= y, \\ n_1\rho - (2np - z)\, q &= z, \end{aligned}\right\} \ \ldots\ldots\ldots (8);$$

and eliminating ρ and q, and substituting for l, m, n from the equation to the surface, we shall find the locus on which the rays (7) were before reflection.

Let $\lambda_1\ \mu_1, \nu_1$ be multipliers; then putting

$$\left.\begin{aligned} \lambda_1 l_1 \quad + \quad \mu_1 m_1 \quad + \quad \nu_1 n_1 \quad &= 0, \\ \lambda_1 (2lp - x) + \mu_1 (2mp - y) + \nu_1 (2np - z) &= 0, \end{aligned}\right\},$$

we get $$\frac{\lambda_1}{2(m_1 n - m n_1)\, p - (m_1 z - n_1 y)} = \frac{\mu_1}{(\quad)} = \frac{\nu_1}{(\quad)};$$

hence, by (8),

$$(m_1 n - m n_1)\, x + (n_1 l - n l_1)\, y + (l_1 m - m l_1)\, z = 0;$$

or if $F(x, y, z) = 0$, be the equation to the surface, and

$$u = \frac{dF}{dx}, \quad v = \frac{dF}{dy}, \quad w = \frac{dF}{dz}, \qquad \text{since } \frac{l}{u} = \frac{m}{v} = \frac{n}{w};$$

therefore the equation to the required locus is

$$(m_1 w - n_1 v)\, x + (n_1 u - l_1 w)\, y + (l_1 v - m_1 u)\, z = 0 \ldots (9).$$

Ex. In the ellipsoid this becomes

$$\left(\frac{m_1 z}{c^2} - \frac{n_1 y}{b^2}\right) x + \left(\frac{n_1 x}{a^2} - \frac{l_1 z}{c^2}\right) y + \left(\frac{l_1 y}{b^2} - \frac{m_1 x}{a^2}\right) z = 0,$$

or $$l_1 yz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + m_1 zx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + n_1 xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0,$$

the equation to a cone.

If in particular $l_1 = m_1 = n_1$, this reduces to

$$yz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + zx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right).$$

Addition.

For *refraction*, when ι' is the angle of refraction, and $\sin \iota = \kappa \sin \iota'$, we get equations (1), (2) unaltered; instead of (3) we have

$$\cos \iota = l\alpha + m\beta + n\gamma, \quad \cos \iota' = l\alpha' + m\beta' + n\gamma',$$
$$\cos(\iota - \iota') = \alpha\alpha' + \beta\beta' + \gamma\gamma';$$

whence it easily follows that

$$-\frac{\lambda}{\sin \iota'} = \frac{\mu}{\sin \iota} = \frac{1}{\sin(\iota - \iota')},$$

or $$l \sin(\iota - \iota') = -\alpha \sin \iota' + \alpha' \sin \iota;$$

and hence equations to the refracted ray

$$\frac{\xi - x}{l \sin(\iota - \iota') + \alpha \sin \iota'} = \frac{\eta - y}{m \sin(\iota - \iota') + \beta \sin \iota'}$$
$$= \frac{\zeta - z}{n \sin(\iota - \iota') + \gamma \sin \iota'} \ldots . (5').$$

These expressions give those for reflection (5) by putting $\kappa = -\iota$; $\iota = -\iota'$, as it ought to be.

L. FISCHER.

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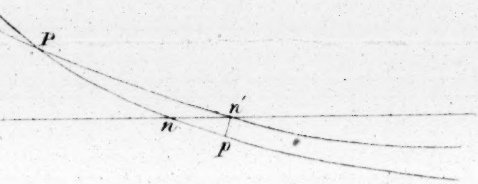

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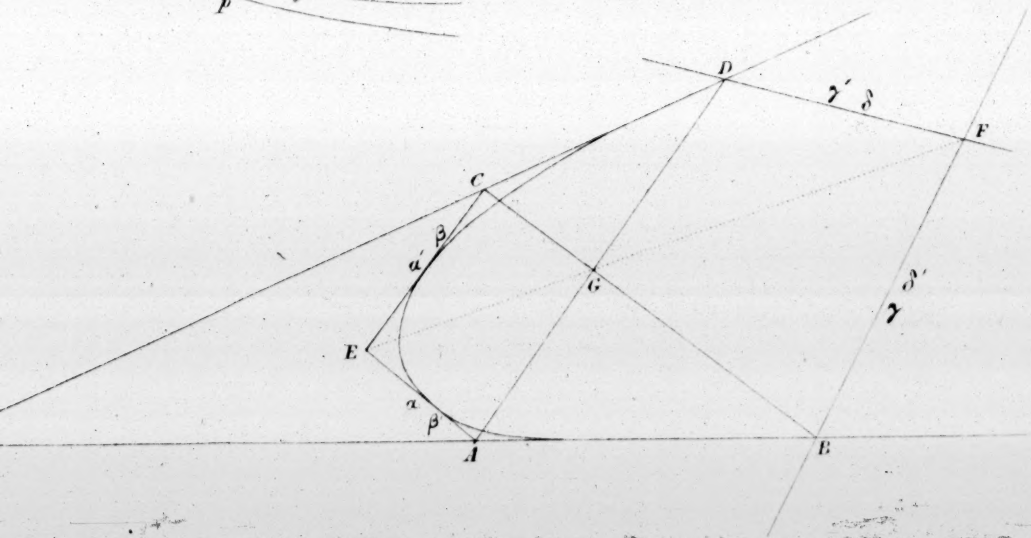

GENERAL INDEX TO THE FIRST SERIES.

GEOMETRY OF TWO DIMENSIONS.

Page

On a property of a triangle—Relation between the radii of inscribed circles. By *Prof. Wallace* . . . vol. i.* 21

Demonstrations of some properties of a triangle—Triangle of least perimeter, &c. By *D. F. Gregory* . . vol. i. 157

On a relation among the angles of polygons inscribed in a circle. vol. i. 192

On the equation to the tangent to the ellipse. By *A. Smith.* vol. i. 9

Evolute to the ellipse . . . . . vol. i. 47

Demonstrations of properties of conjugate diameters in the ellipse vol. i. 95, 96, 239

On some properties of the parabola—Circumscribing hexagon and triangle. By *R. L. Ellis* . . . . vol. i. 204

Demonstrations of some properties of the conic sections—Methods of finding the radius of curvature. By *D. F. Gregory* . vol. i. 61

New analytical methods of solving certain geometrical problems. By *S. S. Greatheed* . . . . . . vol. i. 168

Solutions of some problems in transversals . . vol. i. 87

On some elementary principles in the application of Algebra to Geometry—Interpretation of + and − in Geometry . vol. i. 74

On the existence of branches of curves in several planes. By *D. F. Gregory* . . . . . vol. i. 259

On the elementary principles of the application of algebraical symbols to Geometry. By *D. F. Gregory* . . vol. ii. 1

On a symmetrical form of the equation to the parabola . vol. ii. 14

Demonstrations of expressions for the area of a triangle. By *Prof. Wallace* . . . . . . vol. ii. 35

Problem, "Given the n^{th} part of a straight line to find the $(n+1)^{\text{th}}$ part" . . . . . vol. ii. 47

Demonstrations of two properties of the abscissæ of the angular points of a polygon circumscribing a parabola . . vol. ii. 48

Investigation of the general equation to diametral curves. By *W. Walton* . . . . . vol. ii. 76

On the general theory of the loci of curvilinear intersection. By *W. Walton* . . . . . . vol. ii. 85

On the general interpretation of equations between two variables in algebraic geometry. By *W. Walton* . . vol. ii. 103

On the general theory of multiple points. By *W. Walton* . vol. ii. 155

Simple demonstration of a property of the conic sections. By *A. De Morgan* . . . . . vol. ii. 202

On the existence of real asymptotes to impossible branches of curves. By *W. Walton* . . . . . . vol. ii. 236

On a theorem in the geometry of position. By *A Cayley* vol. ii. 267

Analytical demonstrations of Stewart's theorems. By *R. L. Ellis* vol. ii. 271

Solution of the problem "To find the envelop of a chord of a conic section which subtends a constant angle at the focus." By *J. Booth* vol. iii. 87

* In references to vol. i. the pages of the first edition are indicated.

Page
Demonstrations of some geometrical theorems analogous to that of Cotes. By *D. F. Gregory* . . . vol. iii. 144
On a new property of the ellipse and hyperbola. By *S. Earnshaw* vol. iii. 196
On the intersection of curves. By *A. Cayley* . vol. iii. 211
Symmetrical investigation of points of inflection. By *W. Walton* vol. iv. 13
Demonstration of Pascal's Theorem. By *A. Cayley* . vol. iv. 18
Of asymptotes to algebraical curves. By *D. F. Gregory* vol. iv. 42
Notes on conic sections. By *M. O' Brien* . . vol. iv. 99
On the theory of algebraical curves. By *A. Cayley* . vol. iv. 102
Polar equations to tangents to a conic section. By *P. Frost* vol. iv. 112
On Brianchon's hexagon. By *W. Walton* . . vol. iv. 163
Note on geometrical discontinuity. By *H. Goodwin* . vol. iv. 190
Proof of a property of the parabola . . vol. iv. 192
On Brianchon's hexagon. By *P. Frost* . . vol. iv. 277

GEOMETRY OF THREE DIMENSIONS.

Application of the symmetrical equations of the straight line to various problems in analytical geometry of three dimensions, Nos. 1 & 2. By *S. S. Greatheed* . . . . vol. i. 37, 135
Investigation of the radius of absolute curvature. By *A. Smith* vol. i. 92
Investigation of the existence of circular sections in surfaces of the second order. By *D. F. Gregory* . . . vol. i. 100
Solution of some problems in analytical geometry by means of the symmetrical equations to the straight line . vol. i. 107
Demonstrations of various properties of tangent planes to the ellipsoid vol. i. 117
On certain cases of consecutive surfaces . . vol. i. 187
On the method of spherical coordinates. By *S. S. Greatheed* vol. i. 193
Investigation of the length of the diagonal of a parallelopiped in terms of the edges, and the angles they make with each other. vol. i. 94
On the number of normals which can be drawn from a given point to an algebraical surface. By M. *Terquem* (From *Liouville's Journal*) vol. ii. 11
Investigation of the locus of points for which the sum of the squares of the normals drawn from them to a surface of the second order is constant. By M. *Terquem* (From *Liouville's Journal* vol. ii. 13
Condition that a surface may be touched by a plane in a curved line. By *S. S. Greatheed* . . . . vol. ii. 22
Applications of the method of spherical coordinates. By *S. S. Greatheed*. . . . . . vol. ii. 37
On the perspective of the coordinate planes. By *A. De Morgan* vol. ii. 92
On the lines of curvature in an ellipsoid. By *R. L. Ellis* . vol. ii. 133
Geometrical interpretation of Lagrange's condition in the theory of maxima and minima of functions of two variables. By *D. F. Gregory* . . . . . vol. ii. 138
Symmetrical solutions of problems respecting the straight line and plane. By *George Boole* . . . . vol. ii. 179
On singular points in surfaces. By *D. F. Gregory* . vol. ii. 252
Solution of the problem "To find the envelop of a plane such that the sum of the squares of the perpendiculars on it from n given points is constant." . . . vol. iii. 198
Solution of the problem "To find the equation to a straight line which rests on four given straight lines." . . vol. iii. 231
On the relations between the direction cosines of a line referred to two systems of rectangular coordinates. By *W. Thomson* vol. iii. 248
On a class of differential equations, and on the lines of curvature of an ellipsoid. By *A. Cayley* . . . vol. iii. 264

Page
Proof of a property of the tangent plane to any surface . vol. iv. 47
Demonstration of Dupin's theorem. By *W. Thomson* vol. iv. 62
On the partial differential equations to a family of envelops. By *W. Walton* . . . . vol. iv. 152
Reduction of the general equation of surfaces of the second order. By *W. Thomson* . . . . vol. iv. 227
On the lines of curvature of surfaces of the second order. By *W. Thomson* . . . . vol. iv. 279

ALGEBRA AND TRIGONOMETRY.

On the impossible logarithms of quantities, and the origin of the plurality of roots. By *D. F. Gregory* . . vol. i. 226
Method of elimination by cross multiplication . . vol. i. 46
Demonstrations of propositions in the theory of numbers. By *S. S. Greatheed* . . . . . vol. i. 102, 112
Transformation of the expression $(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2$. By *A. Smith* . . . . vol. i. 155
Solution of certain trigonometrical equations. By *D. F. Gregory* vol. i. 44
New solution of a biquadratic equation. By *S. S. Greatheed* vol. i. 254
On the existence of a relation among the coefficients of the equation of the squares of the differences of the roots of an equation. By *R. L. Ellis* . . . . . vol. i. 256
Algebraical demonstration of the theorem

$$1.2 \,.\, .\, n = n^n - n\,(n-1)^n + \frac{n\,(n-1)}{1.2}\,(n-2)^n + \&c. \quad \text{vol. i. } 239$$

On the residual calculus. By *D. F. Gregory* . . vol. i. 145
On the method of elimination by indeterminate multipliers. By *A. Smith* . . . . . vol. i. 49
On the transformation of the expression $(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2$ vol. ii. 9
Researches in the theory of analytical transformations, with a special application to the reduction of the general equation of the second order. By *George Boole* . . . . vol. ii. 64
Demonstration of a proposition in the theory of numbers. By *S. Earnshaw* . . . . . . vol. ii. 79
Expression for any positive integral power of a logarithm. By *W. Walton* . . . . . vol. ii. 102
On the expansion of sines and cosines of multiple arcs in ascending powers of the cosines and sines of the simple arcs. By. *D. F. Gregory* . . . . . vol. ii. 129
On the expressibility of the roots of algebraic equations. By *J. M. Peebles* . . . . . vol. ii. 145
On the equation of payments. By *W. Walton* . . vol. ii. 188
On the development of the square roots of numbers by continued fractions. By *James Booth* . . . vol. ii. 226
Examples of the dialytic method of elimination as applied to ternary systems of equations. By *J. J. Sylvester* . . vol. ii. 232
Practical rule for reducing a square root into a continued fraction. By *A. De Morgan* . . . . vol. ii. 239
On the irrationality of ε . . . . vol. ii. 240
New solution of a cubic equation. By *J. Cockle* . vol. ii. 248
On a method of algebraic elimination. By *A. Q. G. Craufurd* vol. ii. 276
Exposition of a general theory of linear transformations. By *George Boole*. Parts I. and II. . . . . vol. iii. 1, 106
On the surd roots of equations. By *R. Moon* . . vol. iii. 20
Note on a solution of a cubic equation. By *J. Cockle* . vol. iii. 28
A suggestion in notation. By *A. De Morgan* . . vol. iii. 48
Remarks on the binomial theorem . . . vol. iii. 61

Page

Note on a class of factorials. By *D. F. Gregory* . . vol. iii. 89
Demonstration of Napier's analogies. By *A. De Morgan* . vol. iii. 96
On the general problem of intercalations. By *S. S. Greatheed* vol. iii. 98
Note on the theory of the solutions of cubic and biquadratic equations. By *J. Cockle* . . . . . vol. iii. 104
A method of obtaining any root of a number in the form of a continued fraction . . . . . . vol. iii. 120
Expression for the discontinuous law of the sliding scale in the corn-law By *A. Smith* . . . . vol. iii. 152
On a difficulty in the theory of Algebra. By *D. F. Gregory* vol. iii. 153
On certain expansions in series of multiple sines and cosines. By *A. Cayley* . . . . . vol. iii. 162
On elimination. By *R. Moon* . . . vol. iii. 183
Note on a proposition in the theory of numbers . vol. iii. 247
Mnemonic rule in spherical trigonometry . . vol. iii. 291
Method of finding the greatest common measure of two polynomials. By *A. Q. G. Craufurd* . . . . vol. iv. 9
On the successive approximations of a continued fraction. By *A. De Morgan* . . . . . vol. iv. 97
Chapters in the analytical geometry of (n) dimensions. By *A. Cayley* vol. iv. 119
Notes on linear transformations. By *G. Boole* . . vol. iv. 167
On the theory of linear transformations. By *A. Cayley* vol. iv. 193
Expressions for the sine and cosine of multiple arcs. By *R. Moon* vol. iv. 250

DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.

On general differentiation, Nos. 1 and 2. By *S. S. Greatheed* vol. i. 11, 109
On the expansion of a function of a binomial. Demonstration of a theorem supplemental to that of Taylor. By *S. S. Greatheed* vol. i. 67
Integration of linear differential equations with constant coefficients. By *D. F. Gregory* . . . . . vol. i. 22
Addendum . . . . . . vol. i. 278
Integration of partial differential equations. By *D. F. Gregory* vol. i. 123
Addendum . . . . . . vol. i. 280
Integration of simultaneous differential equations. By *D. F. Gregory* vol. i. 173
Integration of equations of finite and mixed differences. By *D. F. Gregory* vol. i. 54
On the solution of partial differential equations of the form

$$X^a Y^b Z^c \left(\frac{dZ}{dX}\right)^m \left(\frac{dZ}{dY}\right)^n - A = 0$$ vol. i. 162

Demonstrations, by the method of the separation of the symbols, of theorems in the calculi of differentials and of differences. By *D. F. Gregory* . . . . . vol. i. 212
General formulæ for the change of the independent variable. By *S. S. Greatheed* . . . . . vol. i. 236
On the transformation from rectangular to polar coordinates in differential expressions. By *A. Smith* . . vol. i. 121
Note on Fourier's heat—Solution of the equation $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$. By *D. F. Gregory* . . . . . vol. i. 104
Demonstration of Taylor's theorem . . . vol. i. 143
On a property of certain partial differential coefficients—Transformation to rectangular coordinates . . . vol. i. 252
Transformation of homogeneous functions of the second degree. By *A. Smith* . . . . . vol. i. 272
Differences of sin x and cos x . . . vol. i. 238

Page
Decomposition of rational fractions . . . vol. i. 240
Extension of Talbot's theorem to the case of any hyperbola vol. i. 266
On singular solutions and particular integrals of differential equations. By *W. Walton*, Nos. I. and II. . . . vol. ii. 24, 49
On the failure of formulæ in the inverse processes of the differential calculus. By *D. F. Gregory* . . . vol. ii. 73
On certain theorems in the calculus of variations. By *G. Boole* vol. ii. 97
On the integration of linear differential equations with constant coefficients. By *G. Boole* . . . vol. ii. 114
Series for the approximate values of definite integrals. By *A. Smith* vol. ii. 167
On the integration of certain classes of differential equations. By *R. L. Ellis*, Nos. I. and II. . . vol. ii. 169, 193
Investigation of the expression for the radius of curvature in polar coordinates. By *A. Smith* . . . vol. ii. 191
Demonstration of a property of Laplace's coefficients. By *A. Smith* vol. ii. 192
On the integration of equations of partial differentials. By *B. Bronwin* vol. ii. 209
Evaluation of a definite multiple integral. By *D. F. Gregory* vol. ii. 215
Remarks on Poisson's proof that $F(\mu, \omega)$ can be expanded in a series of Laplace's functions. By *H. Goodwin* . . vol. ii. 223
Demonstration of the correctness of Fourier's expansions of functions in series consisting of sines or cosines only. By *W. Thomson* vol. ii. 258
On certain integral transformations. By *B. Bronwin* . vol. ii. 263
Note on the definite integral $\int_0^{\frac{\pi}{2}} \log(\sin\theta)\, d\theta$. By *R. L. Ellis* vol. ii. 282
Theorem respecting general differentials . . vol. ii. 284
Extension of a property of Laplace's functions to homogeneous functions generally. By *A. Smith* . . vol. ii. 283
On the integration of certain differential equations. By *B. Bronwin* . vol. iii. 29, 175
On the properties of a certain symbolical expression. By *A. Cayley* vol. iii. 62
On the limits of Maclaurin's theorem. By *A. Q. G. Craufurd* vol. iii. 84
Addition to the preceding article . . . vol. iii. 159
Remarks on the distinction between algebraical and functional equations. By *R. L. Ellis* . . . . vol. iii. 92
On elliptic functions. By *B. Bronwin* . . vol. iii. 123
Note on the preceding article. By *A. Cayley* . . vol. iii. 197
On the solution of functional differential equations. By *R. L. Ellis* vol. iii. 131
On certain definite integrals. By *Arthur Cayley* . vol. iii. 138
On some definite integrals. By *H. Goodwin* . . vol. iii. 168
Evaluation of certain definite integrals. By *R. L. Ellis* vol. iii. 185
On the transformation of definite integrals. By *George Boole* vol. iii. 216
On certain cases of geometrical maxima and minima . vol. iii. 237
On the solution of certain functional equations. By *D. F. Gregory* vol. iii. 239
On the integration of certain equations in finite differences. By *B. Bronwin* . . . . vol. iii. 258
On partial differential equations. By *W. Walton* . vol. iii. 269
Remarks on a theorem of M. Catalan. By *George Boole* . vol. iii. 277
On Lagrange's theorem. By *A. Cayley* . . vol. iii. 283
On Fourier's theorem. By *D. F. Gregory* . . vol. iii. 288
On certain definite multiple integrals. By *R. L. Ellis* . vol. iv. 1
On the transformation of multiple integrals. By *G. Boole* . vol. iv. 20
Problem, "To find the value of a certain function" . vol. iv. 48
On the equation $(D + a)^n\, y = X$, and Addendum. By *A. De Morgan* vol. iv. 60, 96
Note on a definite multiple integral. By *R. L. Ellis* . vol. iv. 64

Page
On the inverse calculus of definite integrals. By *G. Boole* vol. iv. 82
On a new species of equations of differences. By *A. De Morgan* vol. iv. 87
On a multiple definite integral. By *R. L. Ellis* . vol. iv. 116
On a certain definite integral. By *R. L. Ellis* . . vol. iv. 143
Applications of the symbolical form of Maclaurin's theorem vol. iv. 176
On the use of the symbol $e^{\theta\sqrt{(-1)}}$ in certain transformations. By *H. Goodwin* . . . . vol. iv. 177
Solution of equations in finite differences. By *R. L. Ellis* vol. iv. 182
On the theory of developments. Part I. By *G. Boole* . vol. iv. 214
On certain integral transformations. By *B. Bronwin* . vol. iv. 233
On certain continued fractions. By *P. Frost* . . . vol. iv. 237
On the integrating factor of the homogeneous differential equation of the first order. By *G. G. Stokes* . . vol. iv. 241
On indeterminate maxima and minima. By *W. Walton* . vol. iv. 253
On the inverse elliptic functions. By *A. Cayley* . vol. iv. 257

MECHANICS.

Investigation of the three principal axes of rotation. By *W. Pirie* vol. i. 35
Note on the theory of the spinning-top. By *A. Smith* . vol. i. 42
On the conditions of stable and unstable equilibrium By *J. H. Pratt* vol. i. 131
On the conditions of equilibrium of a rigid system . vol. i. 222
Investigation of the tendency to break of a beam loaded with weights. By *A. Smith* . . . . vol. i. 276
On the oscillations of a floating body. By *S. S. Greatheed* vol. i. 167
On a new form of equilibrium of a revolving fluid (Jacobi's theorem). By *A. Smith* . . . . . vol. i. 90
On the condition of equilibrium of a system of mutually attractive fluid particles. By *R. L. Ellis* . . . vol. ii. 18
On a property of the brachystochrone when the forces are any whatsoever . . . . . . vol. ii. 45
Investigation of the angle between the apsides of the path of a projectile moving in a surface of revolution, when the orbit is nearly circular . . . . . . vol. ii. 62
On the sympathy of pendulums. By *D. F. Gregory & A. Smith* vol. ii. 120
On the tautochrone in a resisting medium. By *R. L. Ellis* vol. ii. 153
On the motion of a pendulum whose point of support is disturbed. By *A. Smith* . . . . . . vol. ii. 204
On the form of a bent spring . . . . . vol. ii. 250
On the motion of a sphere in a cylinder which revolves uniformly in a vertical plane. By *J. Booth* . . vol. iii. 42
On the motion of a particle along variable and moveable tubes and surfaces. By *W. Walton* . . . vol. iii. 49
On the motion of a solid body about its centre of gravity vol. iii. 148
Demonstration of the principle of virtual velocities . vol. iii. 150
Propositions in the theory of attraction. Parts I. and II. By *W. Thomson* . . . . vol. iii. 189, 201
Researches in rotatory motion. By *A. Bell* . vol. iii. 213
On the motion of rotation of a solid body. By *A. Cayley* vol. iii. 224
Note on a problem in dynamics . . . vol. iii. 267
On the equations of motion of rotation. By *A. Bell* . vol. iv. 7
On the motion of a piston and of the air in a cylinder. By *G. G. Stokes* vol. iv. 28
On the motion of the centre of gravity of a broken body. By *W. Walton* vol. iv. 77
On the axis of spontaneous rotation. By *H. Goodwin* . vol. iv. 158
Note on the spontaneous axis of rotation. By *W. Walton* . vol. iv. 173
On the law of gravity at the surface of a revolving homogeneous fluid. By *W. Thomson* . . . . vol. iv. 191

ASTRONOMY.

Page
Analytical solutions of problems in plane astronomy. By *S. S. Greatheed* . . . . . . vol. i. 182
Method of investigating $\frac{d\theta}{de}$ and $\frac{dr}{de}$ in the planetary theory. By *W. Pirie* vol. i. 47
On the variation of the longitude of the perihelion in the planetary theory. By *M. O'Brien* . . . vol. i. 165
Investigation of the Moon's annual and secular inequalities, so far as they depend on the variation of the central force. By *A. Smith* vol. i. 64
Investigation of the general term of the expansion of the true anomaly in terms of the mean. By *S. S. Greatheed* . . vol. i. 208
Investigation of the equation which determines the stability of the planetary excentricities. . vol. i. 234
Investigations of the variations of the node and inclination . vol. ii. 113
Geometrical investigation of the aberration in right ascension and declination. By *R. L. Ellis* . . . vol. ii. 120
Analytical solutions of problems in plane astronomy . vol. ii. 178
On the relation between the long inequalities of two mutually disturbing planets . . . . vol. iii. 95
On certain formulæ made use of in physical astronomy. By *M. O'Brien* vol. iii. 249
On the stability of the eccentricities and inclinations of the planetary orbits. . . . . . vol. iii. 290
On the lunar theory. By *M. O'Brien* . . vol. iv. 49
On a problem in precession and nutation. By *W. Walton* . vol. iv. 137
On a problem in central forces. By *W. Walton* . vol. iv. 171
Transformation of the differential equations of a planet's motion. By *B. Bronwin* . . . . . vol. iv. 245

LIGHT AND SOUND.

Notes on the wave theory of light. Nos. 1 and 2. By *A. Smith* vol. i. 1, 77
On the achromatism of eye-pieces of telescopes and of microscopes. By *R. L. Ellis* . . . . . vol. i. 269
On the propagation of a wave in an elastic medium. By *A. Smith* vol. i. 97
On the vibration of a musical string. By *J. Power* . vol. i. 241
Method of finding the approximate values of definite integrals which occur in the calculations of diffraction . . vol. ii. 81
Note on the calculation of the intensity of light at the centre of the shadow of a small circular disk. By *A. Smith* . vol. ii. 141
On fringes of interference produced by oblique reflection at the surface of a small mirror. By *A. Bell* . . . vol. ii. 245
On the determination of the intensity of vibration of wavelets diverging from every point of a plane wave. By *A. Smith* vol. iii. 46
Notes on some points in formal optics. By *A. Smith* . vol. iii. 122
Note on the measure of intensity in the wave theory of light. By M. *Abria* (from *Liouville's Journal*) . . vol. iii. 200
Note on vibrating cords. By *W. Thomson* . . vol. iii. 257
On the intensity of light in the shadow of a very small circular disc. By *H. Goodwin* . . . . vol. iv. 72
Demonstration of a proposition in physical optics . vol. iv. 115
Note on a problem in plane optics. By *L. Fischer* . vol. iv. 286

HEAT, ELECTRICITY, AND MAGNETISM.

Page

Note on a passage in Fourier's theory of heat. By *W. Thomson* vol. iii. 25

On the uniform motion of heat in solid bodies, and its connection with the mathematical theory of electricity. By *W. Thomson* vol. iii. 71

On the linear motion of heat. Parts I. and II. By *W. Thomson* vol. iii. 170, 206

On the attractions of conducting and non-conducting electrified bodies. By *W. Thomson* . . . . . . vol. iii. 275

On orthogonal isothermal surfaces. By *W. Thomson* . . vol. iii. 286

On the equations of the motion of heat referred to curvilinear coordinates. By *W. Thomson* . . . vol. iv. 33

On some points in the theory of heat By *W. Thomson* . vol. iv. 67

Notes on Magnetism. Parts I. and II. By *R. L. Ellis* vol. iv. 90, 139

Note on orthogonal isothermal surfaces. By *W. Thomson* vol. iv. 179

Demonstration of a fundamental proposition in electricity. By *W. Thomson* . . . . vol. iv. 223

MISCELLANEOUS.

Solutions of problems in the Senate-house papers. vol. i. 32, 95, 144, 282

Solutions of Senate-house problems . vol. ii. 94, 142, 143

Solutions of problems . . . vol. iii. 95, 152, 292

On the knight's move at chess. By *R. Moon* . vol. iii. 233

Memoir of D. F. Gregory, M.A., Fellow of Trinity College. By the *Editor* . . . . . . vol. iv. 145

On a question in the theory of probabilities. By *R. L. Ellis* vol. iv. 127

On the balance of the chronometer. By *R. L. Ellis* . vol. iv. 133

On magic squares. By *R. Moon* . . . vol. iv. 209